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Bifurcations of Central Configurations in the N-Body Problem

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ABSTRACT. This paper discusses a series of studies done by the authors on the bifurcations of central configurations in the N-body problem. Modern bifurcation analysis and algebraic processors like the general purpose processor MACSYMA and the special purpose processor POLYPAK were used to find a multitude of different bifurcations.

I Introduction. The study of central configurations (c.c.) of the N-body problem has had a long history starting with the famous collinear configuration of the 3-body problem found by Euler (1767). Over the intervening years many different technologies have been applied to the study of c.c. In the older papers of Euler (1767), Lagrange (1772), Hoppe (1879), Lehmann-Filhes (1891), Moulton (1910) et al. special coordinates, symmetries and analytic techniques were used. Dziobek (1900) used the theory of determinants; Smale (1970) used Morse theory; Palmore (1975) used homology theory; Simo (1977) used a computer; and Moeckel (1986) used real algebraic geometry in their investigations. Thus, the study of c.c. has been a testing ground for many different methodologies of mathematics.

In a series of papers, Meyer(1987), Meyer and Schmidt (1988a,1988b), Schmidt(1988), we have used the methods of modern bifurcation analysis and automated algebraic processor to study this subject. Specifically, in the first paper, Meyer(1987), a fold catastrophe or saddle-node type bifurcation was established in the four body problem by continuing a bifurcation in the restricted four body problem into the full four body problem with one small mass. The point of bifurcation was found using numerical methods and then established by rigorous analysis. In the second and third papers, Meyer and Schmidt (1988a,1988b), the bifurcations of a central configuration which consists of N-1 particles of mass 1 at the vertices of a regular polygon

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and one particle of mass m at the centroid was studied. We call this the regular polygon central configuration (r.p.c.c.). In the second paper, we considered the 4 and 5 body problems and use the mutual distances as special coordinates following the lead of Dziobek(1900). These coordinates make the 4-body problem relatively easy to handle and the 5-body problem accessible, but beyond 5, Dziobek's coordinates become very cumbersome. The 4 and 5-body problem in these special coordinates are sufficiently simple that the general purpose algebraic processor MACSYMA could handle the tedious calculations. In the third paper the investigation of the bifurcations of these c.c. for larger N required the special purpose algebraic processor POLYPAK written by the second author because the computations increased rapidly with N. In the analysis of the 4 and 5 body problems the classical power series methods of bifurcation analysis handles the problem nicely, but for larger N a systematic use of Lie transforms by Deprit (1969) was mandated in order to bring the equations into a normal form. The first three papers dealt with the planar N-body problem, but in Schmidt(1988) the Dziobek coordinates were used with the aid of MACSYMA to find a bifurcation from a tetrahedron configuration of the spatial 5-body problem.

The problem of finding a c.c. can be reduced to finding a critical point of the potential energy function on the manifold of constant moment of inertia. Thus the problem falls within the domain of catastrophe theory and so the general theory is well understood. However, this specific problem has a high degree of symmetry, many variables and a constraint, so the computations must be performed with care. We consider these papers as case studies in bifurcation analysis in' face of these complexities.

Even though as solutions of the N-body problem c.c. are quite rare and rather special, they are of central importance in the analysis of the asymptotic behavior of the universe. In general, solutions which expand beyond bounds or collapse in a collision do so asymptotically to a central configuration. A survey and entrance to this literature can be found in Saari (1980).

II Central Configurations for the N-body. The N-body problem is the system of differential equations which describe the motion of N particles moving under the influence of their mutual gravitational attraction. Let $q_j \in \mathbb{R}^3$ be the position vector, $p_j \in \mathbb{R}^3$ the momentum vector and $m_j > 0$ the mass of the j^{th} particle, $1 \le j \le N$, then the equations of motion are

$$(1)\qquad \begin{aligned} \dot{q}_j &= \frac{\partial H}{\partial p_j} = \frac{1}{m_j} p_j, \\ \dot{p}_j &= -\frac{\partial H}{\partial q_j} = \frac{\partial U}{\partial q_j} \end{aligned}$$

where H is the Hamiltonian

$$(2)\qquad H = \sum_{j=1}^{N} \frac{\| p_j \|^2}{2m_j} - U(q)$$

and U is the (self) potential

$$(3)\qquad U = \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{\| q_i - q_j \|}.$$

These equations reduce to the Newtonian formulation

$$(4)\qquad m_j \ddot{q}_j = \frac{\partial U}{\partial q_j}, \quad j = 1,\ldots,N.$$

We seek a homothetic solution by setting $q_j = \phi(t) u_j$ where ϕ is a scalar function and the u_j are constants. Such a solution exists provided there is a constant λ so that the following equations are satisfied.

$$(5)\qquad \ddot{\phi} = \frac{-\lambda}{\phi^2}$$

$$(6)\qquad -\lambda m_j u_j = \frac{\partial U}{\partial u_j}, \; j = 1,\ldots,N.$$

Equation (5) is just the differential equation of the collinear Kepler problem and so has many solutions. Equation (6) is an algebraic equation for the N vectors $u_1,\ldots,u_N$ and the scalar λ. If $u_1,\ldots,u_N$ satisfy (6) for some λ then $u_1,\ldots,u_N$ is called a *central configuration.*

It is classical and easy to verify that if $\bar{u} = (\bar{u}_1,\ldots,\bar{u}_N)$ and $\bar{\lambda}$ is a solution of (6) then the center of mass of $\bar{u}$ is at the origin ($\sum m_j \bar{u}_j = 0$) and $\bar{\lambda} = U(\bar{u})/2I(\bar{u}) > 0$ where I is the moment of inertia

$$(7)\qquad I(u) = \frac{1}{2} \sum_{j=1}^{N} m_j \| u_j \|^2.$$

We will set

$$M = \{ u \in \mathbb{R}^{3N} : \sum m_j u_j = 0 \}$$

(8) $$\Delta = \{ u \in \mathbb{R}^{3N} : u_i = u_j \text{ for some } i \neq j \}$$

$$S = \{ u \in M : I(u) = 1 \}.$$

The variable λ can be considered as Lagrange multiplier and so an equivalent definition of a central configuration is a critical point of U restricted to $S \setminus \Delta$. If u is a c.c. then so is $Au = (Au_1 ,\dots, Au_N)$ where $A \in O(3,\mathbb{R})$ is an orthogonal matrix. We can define an equivalence relation by $u \sim Au$ when $A \in O(3,\mathbb{R})$ and since U, I, are constant on equivalence classes we can define the quotient spaces $\mathcal{S} = (S \setminus \Delta) / \sim$ and the function $\mathcal{U} : \mathcal{S} \to R$ by $\mathcal{U}([u]) = U(u)$ where [ ] denotes an equivalence class. $\mathcal{S}$ and $\mathcal{U}$ are smooth. Thus a similarity class of c.c. is a critical point of $\mathcal{U}$.

A central configuration is called non-degenerate if its equivalence class is a non-degenerate critical point of $\mathcal{U}$ in the sense of Morse theory, i.e. the Hessian is non-singular at the critical point. It follows from the implicit function theorem that bifurcations can occur only at degenerate critical points, so the first quest is to find degenerate c.c.

III. The Restricted Problem. Consider the planar (N+1)-body problem with one particle small, so let $m_{N+1} = \varepsilon$ and $x = u_{N+1}$. The equations for a c.c. become

(1)
$$-\lambda m_j u_j = \frac{\partial U_N}{\partial u_j} + O(\varepsilon) \text{ , } j = 1,\dots,N$$
$$-\lambda x = \frac{\partial W}{\partial x},$$

where $W = \sum_{i=1}^{N} (m_i/\|x-u_i\|)$ and U_N is the self potential of the N-body problem. When $\varepsilon = 0$ the equations in (1) decouple and a solution is an (N+1)-tuple $(\bar{u}_1,\dots,\bar{u}_N,\bar{x})$ where $\bar{u}_1,\dots,\bar{u}_N$ is a c.c. for the N-body problem and $\bar{x}$ is a critical point of

(2) $$V(x) = \sum_{i=1}^{N} \frac{m_i}{\| x - \bar{u}_i \|} + \frac{\lambda}{2} \|x\|^2.$$

Note that λ is determined by the fact that $\bar{u}_1,\dots,\bar{u}_N$ is a c.c. The function V is called the potential of the restricted N-body problem.

It is not too hard to verify that if $\bar{u}_1,\dots,\bar{u}_N$ is a nondegenerate c.c. and $\bar{x}$ is a nondegenerate critical point of (2) then for small ε the full (N+1)-body problem has a nondegenerate c.c close to $(\bar{u}_1,\dots,\bar{u}_N,\bar{x})$. In Meyer(1987) it was proven that there is a degenerate c.c. in the full 4-body problem by showing that the restricted 4-body problem has a fold catastrophe and this fold catastrophe can be continued into the full 4-body problem.

Consider the one parameter family of the restricted 4-body problems where the c.c. of the 3-body problem is the equilateral triangle c.c. with

$$\begin{aligned} m_1 &= 1 - \mu, & \bar{u}_1 &= (1, -\sqrt{3}\mu), \\ m_2 &= 1 - \mu, & \bar{u}_2 &= (-1, -\sqrt{3}\mu), \\ m_3 &= 2\mu, & \bar{u}_3 &= (0, \sqrt{3}(1-\mu)). \end{aligned} \tag{3}$$

For $\mu \simeq 0.4234$ this potential has a fold catastrophe, i.e. a critical point with $V_1 = V_2 = V_{12} = V_{22} = 0$ and $V_{11} \neq 0$, $V_{222} \neq 0$ and $V_{23} \neq 0$ where the subscripts 1,2,3 denote differentiation with respect to x_1, x_2 and μ respectively. Figure 1 shows the potential for $\mu = 0.2$; note the two minima and the saddle point in front of the middle pole. Figure 2 show the potential for $\mu = 0.5$; note that there is only one minimum in front of the middle pole and that one minimum and the saddle point are gone. The viewer for the three dimensional plots in Figures 1 and 2 is situated above the negative x_2-axis at a point that is on a line that makes a 60^0 with the V-axis. At the fold, the saddle point and one minimum come together and eliminate each other. A careful application of the implicit function theorem shows that this fold catastrophe persists in the full 4-body problem for ε small.

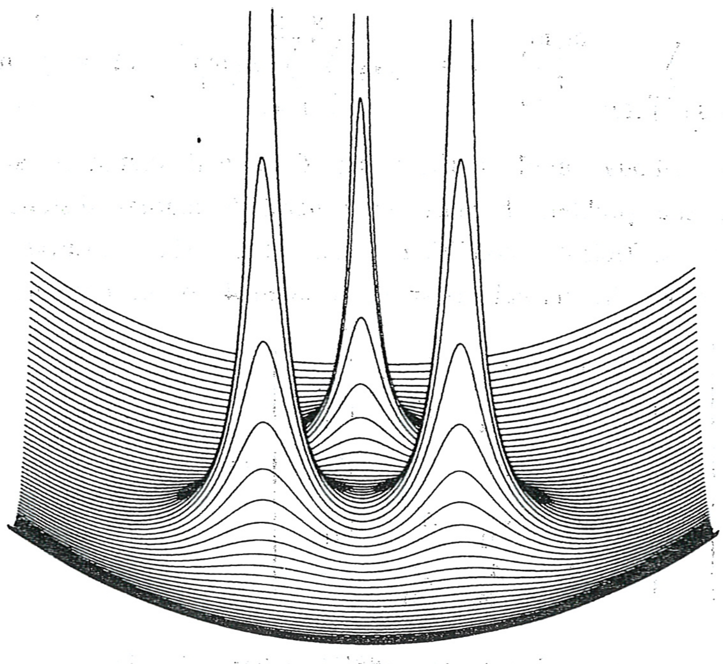


Figure 1. The potential V for $\mu = 0.2$.

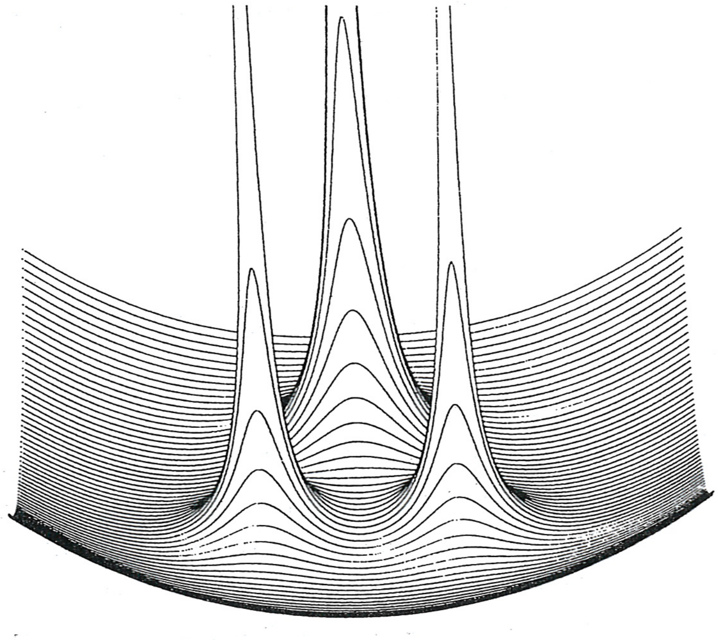


Figure 2. The potential V for $\mu = 0.5$.

IV. Dziobek Coordinate. Dziobek(1900) used the mutual distances $r_{ij} = \| u_i - u_j \|$ as coordinates in his study of c.c. in the planar 4-body problem. The potential, U, and the moment of inertia, I, can easily be expressed in terms of the mutual distances, in fact

$$(1) \qquad U = \sum_{1 \le i<j \le N} \frac{m_i m_j}{r_{ij}}, \quad I = \frac{1}{4M} \sum_{i=1}^{N} \sum_{j=1}^{N} m_i m_j r_{ij}^2, \quad M = \sum_{i=1}^{N} m_i.$$

For the planar 4-body problem there are 6 mutual distances and clearly they over determine the problem because in general 5 mutual distances suffice. A necessary and sufficient condition that the six positive numbers r_{ij}, $1 \le i < j \le N$ be the mutual distances between 4 collinear points is

$$(2) \qquad F = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & r_{12}^2 & r_{13}^2 & r_{14}^2 \\ 1 & r_{12}^2 & 0 & r_{23}^2 & r_{24}^2 \\ 1 & r_{13}^2 & r_{23}^2 & 0 & r_{34}^2 \\ 1 & r_{12}^2 & r_{24}^2 & r_{34}^2 & 0 \end{vmatrix} = 0.$$

This 19th century determinant is $288V^2$ where V is the volume of the tetrahedron whose 6 edges are given. $F = 0$ is simply another constraint.

Thus we follow Dziobek to find c.c. by studying the critical points of

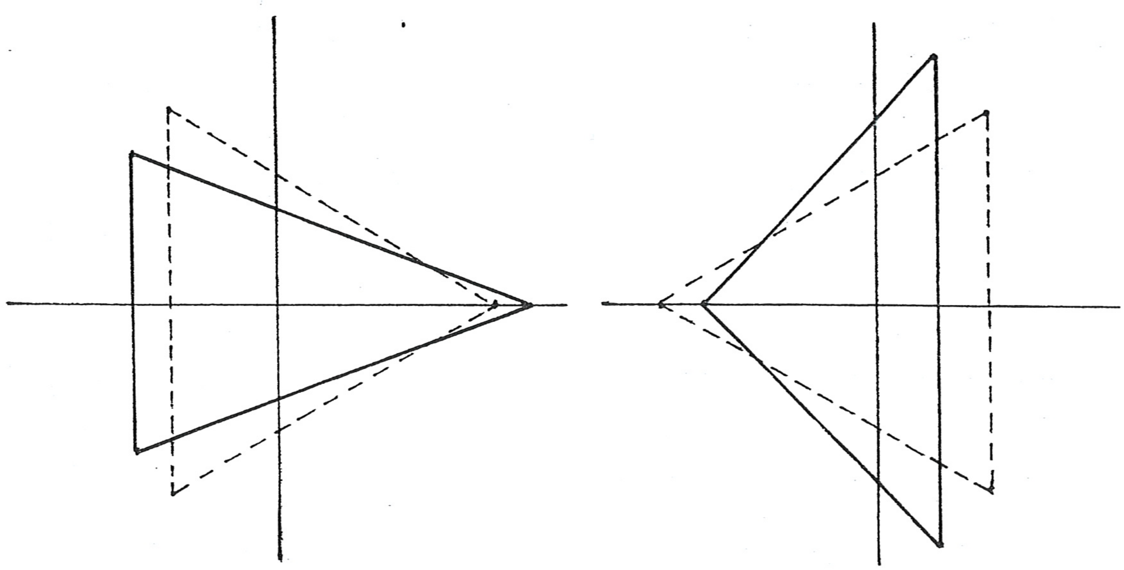
(3) $\qquad W = U + \lambda I + \upsilon F,$

where λ and υ are Lagrange multipliers.

Consider the one parameter family of c.c. in the 4-body problem where 3 of the particles have mass 1 and are located at the vertices of an equilateral triangle and the fourth particles has mass m and is located at the centroid of the triangle. Palmore(1973) showed that for $m = m^* = (64\sqrt{3}+81)/249$ this c.c. is degenerate. We verify Palmore's result in Meyer and Schmidt(1988a) and also show that m^* is a point of bifurcation. For $m < m^*$ there is an acute isosceles triangle c.c. which approaches the equilateral family as $m \to m^*-$, see Figure 3a. For $m > m^*$ there is an obtuse isosceles triangle c.c. which approaches the equilateral family as $m \to m^*+$, see Figure 3b.

The computations are fairly lengthy and so MACSYMA was used to carry out the details. This method was also used in Meyer and Schmidt(1988a) to find a bifurcation in the planar 5-body problem which is similar to the bifurcation discussed above.

The Dziobek coordinates where used in Schmidt(1988) to find a bifurcation of the spatial 5-body problem. Consider a c.c. where 4 particles of mass 1 are placed at the vertices of a regular tetrahedron and one particle of mass m is placed at the centroid. For $m = m^{\#} = (10368+1701\sqrt{6})/54952$ this c.c. is degenerate and is also a point of bifurcation. A family of tetrahedron bifurcates from the regular tetrahedron in a manner similar to the bifurcation of the triangles in the planar problem.



3a. $m < m^*$ 3b. $m > m^*$

Figure 3.

V. Large N. In Meyer and Schmidt(1988b) the bifurcations of the regular polygon central configuration for large N was investigated using different methods and a different algebraic processor. The number of unknowns and equations increase with N. Also due to the high degree of symmetry in the problem for large N, knowledge of very high order terms in the series is required in order to determine the nature of the bifurcation. Therefore, a systematic use of normal form theory, the method of Lie transforms of Deprit(1968), and the special purpose algebraic processor **POLYPAK** written by the second author was necessary.

For N large there are more and more critical values of m which make the r.p.c.c. degenerate and all the cases that we investigated gave rise to a bifurcation and hence to new c.c. Figure 4 shows four of the eleven bifurcations which occur for the 13-body problem (12 around a regular 12-agon and 1 at the centroid).

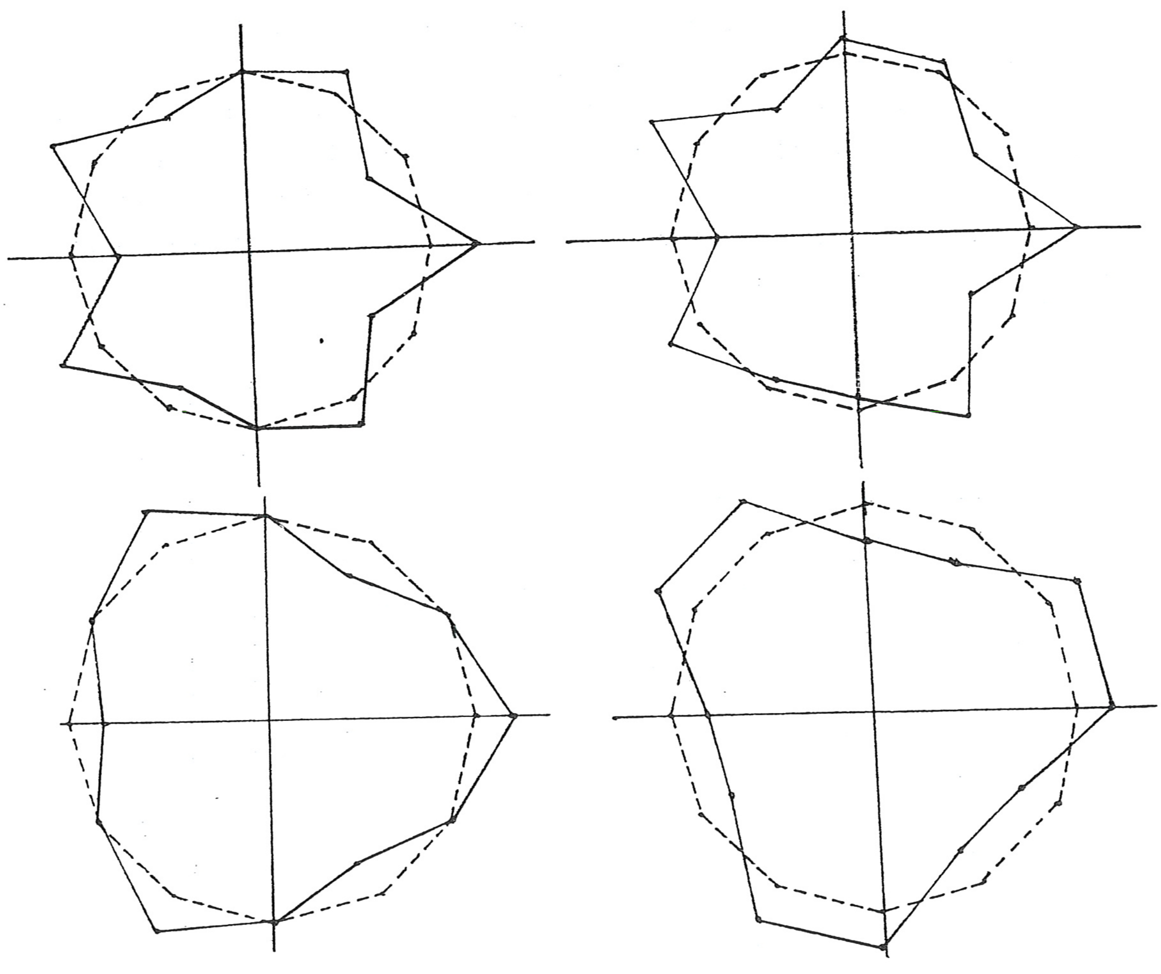


Figure 4. Some bifurcations in the 13-body problem.

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