THE EVOLUTION OF THE STABLE AND UNSTABLE MANIFOLD OF AN EQUILIBRIUM POINT

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Abstract. We consider the evolution of the stable and unstable manifolds of an equilibrium point of a Hamiltonian system of two degrees of freedom which depends on a parameter, ν. The eigenvalues of the linearized system are complex for $\nu < 0$ and purely imaginary for $\nu > 0$. Thus for $\nu < 0$ the equilibrium has a two-dimensional stable manifold and a two-dimensional unstable manifold, but for $\nu > 0$ these stable and unstable manifolds are gone. We study the system defined by the truncated generic normal form in this situation.

One of two things happens depending on the sign of a certain quantity in the normal form expansion. In one case the two families detach as a single invariant manifold and recedes from the equilibrium as ν tends away from 0 through positive values. In the other case the stable and unstable manifold are globally connected for $\nu < 0$ and the whole structure of these manifolds shrinks to the equilibrium as $\nu \to 0$ and disappears.

These considerations have interesting implications about Strömgren's conjecture in celestial mechanics and the blue sky catastrophe of Devaney.

Key words: stable manifold, bifurcation, restricted three-body problem, Strömgren's conjecture.

1. Introduction

We consider a Hamiltonian system of two degrees of freedom which depends on a single parameter ν and which has an equilibrium point at the origin for all values of the parameter. The linearization of this system at the origin has a coefficient matrix $A(\nu)$ which is a 4×4 Hamiltonian matrix, so its eigenvalues are symmetric with respect to both the real and imaginary axis [11]. We are interested in the case when the eigenvalues change from complex numbers of the form $\pm\alpha \pm \beta\mathrm{i}$, $\alpha, \beta \neq 0$ when $\nu < 0$ to two pairs of pure imaginary eigenvalues of the form $\pm\omega_1\mathrm{i}$, $\pm\omega_2\mathrm{i}$, ω_1, $\omega_2 \neq 0$ when $\nu > 0$. Clearly $A(0)$ must have a single pair of pure imaginary eigenvalues of multiplicity two, that is, its eigenvalues are of the form $\pm\omega\mathrm{i}$, $\pm\omega\mathrm{i}$.

Much is known about the local geometry of the flow in the two cases when $\nu < 0$ and $\nu > 0$. For example when $\nu < 0$ the origin is unstable and when $\nu > 0$ the origin is linear stable and sometimes Arnold's theorem [1] implies stability. Also when $\nu < 0$ the equilibrium point is a saddle point with two-dimensional stable and unstable manifolds [5], but when $\nu > 0$ the Liapunov Center Theorem [9] assures that two families of periodic solutions emanate from the equilibrium point. How do these structures change?

In 1971, Meyer and Schmidt [12] stated and proved the theorem that has become known as the Hamiltonian–Hopf Theorem which tells what happens to the Liapunov families of periodic solutions provided a certain quantity η is nonzero. The quantity η depends on the normal form of A at $\nu = 0$ and on a particular term in the normal form expansion of H and it will be defined below. In the Case A when $\eta < 0$ the two Liapunov families are globally connected for $\nu > 0$ and shrink to the equilibrium as $\nu \to 0^+$. In the Case B when $\eta > 0$ the two Liapunov families detach from the equilibrium as a single family as ν decreases from zero. Meyer and Schmidt [12] using a computation of Henrard and Deprit [3] show that the $\eta > 0$ in the restricted three-body problem at the $\mathcal{L}_4$ with $\nu = \mu_1 - \mu$ where μ is the mass ratio parameter and μ_1 is Routh's critical mass ratio parameter. Thus they prove that in the restricted problem the two Liapunov families detach as a unit and recede from $\mathcal{L}_4$ as μ increases through μ_1.

In this paper we shall do a similar formal study of the evolution of the stable and unstable manifolds. Superficially, the story sounds the same with the sign of η reversed. In the Case A when $\eta < 0$ the stable and unstable manifolds detach from the equilibrium as a single invariant manifold as ν increases from zero. In the Case B when $\eta > 0$ the stable and unstable manifolds are globally connected for $\nu < 0$ and shrink to the equilibrium as $\nu \to 0^-$. A rigorous local analysis of the evolution of these manifolds for a complete system which includes un-normalized higher order terms will appear in McSwiggen and Meyer [10].

2. The System of Equations

Consider a Hamiltonian system of two degrees of freedom which depends on a parameter ν which has an equilibrium point at the origin for all ν. That is, a system of the form

$$\dot{z} = J\nabla_z H(z, \nu) = A(\nu)z + F(z, \nu) \tag{1}$$

where $z \in \mathbb{R}^4$, $t, \nu \in \mathbb{R}$, H: $\mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}$ is smooth, J is the 4×4 skew symmetric matrix

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$A(\nu) = J\partial^2 H/\partial z^2(0, \nu)$, $F(z, \nu) = J\nabla_z H(z, \nu) - A(\nu)z$ and $\dot{} = \mathrm{d}/\mathrm{d}t$. Since the equilibrium point is at the origin $\nabla_z H(0, \nu) = F(0, \nu) = 0$ and since $A(\nu)$ is the linear part of the equation $\partial F(0, \nu)/\partial z\,(0, \nu) = 0$. The basic assumption is that when $\nu = 0$ the matrix A has eigenvalues $\pm\omega\mathrm{i}$ of multiplicity two and as ν decreases from zero these eigenvalues move off the imaginary axis.

Because of the complexity of this problem we shall not consider the general case at this time, but consider the truncated normal form in the generic case (see [10] for the analytic details). That is, we assume that H is composed of the first few terms in Sokol'skii's normal form [14, 11].

Sokol'skii's normal form depends on the quantities

$$\begin{aligned} \Gamma_1 &= x_2 y_1 - x_1 y_2, & \Gamma_2 &= \tfrac{1}{2}(x_1^2 + x_2^2), \\ \Gamma_3 &= \tfrac{1}{2}(y_1^2 + y_2^2), & \Gamma_4 &= x_1 y_1 + x_2 y_2, \end{aligned} \tag{2}$$

where $z = (x_1, x_2, y_1, y_2)$. The Hamiltonian (1) is in Sokol'skii's normal form if

$$H = \omega\Gamma_1 + \delta\Gamma_2 + \nu\delta\Gamma_3 + H^{\dagger}(\Gamma_1, \Gamma_3, \nu), \tag{3}$$

where $H^{\dagger}$ is at least quadratic in Γ_1, Γ_3 or in ν and $\delta = \pm 1$.

To see which terms are the most important near the origin and when ν is small we will use the scaling in [12] which was used to identify the important terms for the Hamiltonian–Hopf bifurcation. Scale the variables by

$$\begin{aligned} & x_1 \to \epsilon^2 x_1, && x_2 \to \epsilon^2 x_2, \\ & y_1 \to \epsilon y_1, && y_2 \to \epsilon y_2, \\ & \nu \to \epsilon^2 \nu, \end{aligned}$$

which is symplectic with multiplies ϵ^3. The Hamiltonian becomes

$$H = \omega\Gamma_1 + \epsilon\{\delta\Gamma_2 + \nu\delta\Gamma_3 + \eta\delta\Gamma_3^2\} + \mathrm{O}(\epsilon^2).$$

This indicates that the most important terms are those displayed and so we shall consider the system with only those terms where all the coefficients are nonzero. Thus, we shall investigate the system

$$H = \Gamma_1 + \delta\Gamma_2 + \nu\delta\Gamma_3 + \eta\delta\Gamma_3^2. \tag{4}$$

We have set $\omega = 1$ which can be accomplished by a change of time. By the theory of normal forms for Hamiltonian matrices we may assume that $\delta = \pm 1$ ([7, 15], also see [11]). The unfolding parameter is ν and η is the coefficient of the only nonlinear term in the equations of motion. These are the important terms in the unfolding of a Hamiltonian matrix with a multiple pure imaginary eigenvalue. A more complete discussion of the truncated system with a different objective can be found in [13].

The linearized equations are obtained by setting $\eta = 0$ and so the linearized equations $\dot{z} = A(\nu)z$ has a coefficient matrix

$$A(\nu) = \begin{pmatrix} 0 & 1 & \nu\delta & 0 \\ -1 & 0 & 0 & \nu\delta \\ -\delta & 0 & 0 & 1 \\ 0 & -\delta & -1 & 0 \end{pmatrix}. \tag{5}$$

The eigenvalues are

$$\lambda = \pm \mathrm{i}\sqrt{1 \pm 2\sqrt{\nu} + \nu} = \pm \mathrm{i}(1 \pm \sqrt{\nu}). \tag{6}$$

Thus for small ν, the eigenvalues are complex when $\nu < 0$ and pure imaginary when $\nu > 0$.

The Γ's are the natural invariants of this system and they satisfy a simple system of differential equations, namely

$$\begin{aligned} \dot{\Gamma}_1 &= 0, & \dot{\Gamma}_2 &= \nu\delta\Gamma_4 + \eta\delta\Gamma_3\Gamma_4, \\ \dot{\Gamma}_3 &= -\Gamma_4, & \dot{\Gamma}_4 &= -2\Gamma_2 + 2\nu\delta\Gamma_3 + 2\eta\delta\Gamma_3^2. \end{aligned}$$

We could study this system, but the geometry is unfamiliar, and so we shall follow Sokol'skii and use polar coordinates. We know the singularities and pitfalls of polar coordinates well.

3. Polar Coordinate Form of Equations

Specifically, make the symplectic change of coordinates

$$\begin{aligned} x_1 &= R\cos\theta - \frac{\Theta}{r}\sin\theta, & y_1 &= r\cos\theta, \\ x_2 &= R\sin\theta + \frac{\Theta}{r}\cos\theta, & y_2 &= r\sin\theta \end{aligned}$$

with inverse

$$\begin{aligned} r &= \sqrt{y_1^2 + y_2^2}, & R &= \frac{x_1 y_1 + x_2 y_2}{r}, \\ \theta &= \tan^{-1}\frac{y_1}{y_2}, & \Theta &= x_2 y_1 - x_1 y_2. \end{aligned}$$

The Hamiltonian (3) becomes

$$H = \Theta + \frac{\delta}{2}\left\{R^2 + \frac{\Theta^2}{r^2}\right\} + \frac{\nu\delta}{2}r^2 + \frac{\eta\delta}{4}r^4, \tag{7}$$

and the equations of motion become

$$\begin{aligned} \dot{\theta} &= 1 + \frac{\delta\Theta}{r^2}, & \dot{\Theta} &= 0, \\ \dot{r} &= R, & \dot{R} &= \frac{\delta\Theta^2}{r^3} - \nu\delta - \eta\delta r^3. \end{aligned} \tag{8}$$

From the above we see that θ is an ignorable coordinate and its conjugate momentum Θ is an integral. Thus, we can set $\Theta = c$ where c is an arbitrary constant and ignore θ at least temporarily. The usual convention of polar coordinates hold; in particular, θ is arbitrary, so for fixed r, R, we have a circle if $r \neq 0$ or a point if $r = 0$. We must first study the one degree of freedom problem defined by

$$H = c + \frac{\delta}{2}\left\{R^2 + \frac{c^2}{r^2}\right\} + \frac{\nu\delta}{2}r^2 + \frac{\eta\delta}{4}r^4. \tag{9}$$

This is the Hamiltonian of the second order system

$$\ddot{r} - \frac{c^2}{r^3} + \nu r + \eta r^3 = 0.$$

Thus, the analysis is reduced to the elementary plotting of the level curves of (9), but unfortunately there are three parameters to contend with.

Since the stable and unstable manifolds lie in the $H = 0$ level set, we shall only consider the flow on this level set. The phase portraits for other values of H are easily obtained. In (9), set $H = 0$ and solve for R^2 to obtain

$$R^2 = -2c\delta - \frac{c^2}{r^2} - \nu r^2 - \frac{1}{2}\eta r^4. \tag{10}$$

Fixing ν, δ and η fixes the parameters in the equation, then r, R, θ, and c sweep out the level set where $H = 0$. One need only plot the graph of R^2 for various values of the parameter and then take the square root of the graphs. But since we are only interested in the stable and unstable manifolds we need only consider the level set where $c = 0$.

There are two cases depending on the sign of η. Case A when $\eta < 0$ is illustrated in Figure 1 and Case B when $\eta > 0$ is illustrated in Figure 2.

Recall that these are illustrations of projections of the $H = 0$ level set onto the r, R-plane, and that θ is arbitrary. Over each point (r, R) with $r \neq 0$ there is a circle in $H = 0$, but these circles tend to zero as $r \to 0^+$, and above each point where $r = 0$ there is just a single point. Thus, in Figure 1 for example, there is a curve emanating from the origin. Above the origin is a point and above all the other points on the curve is a circle. Thus this curve represents a plane in $H = 0$ – the unstable manifold. These figures verify the statements about the evolution of the stable and unstable manifolds.

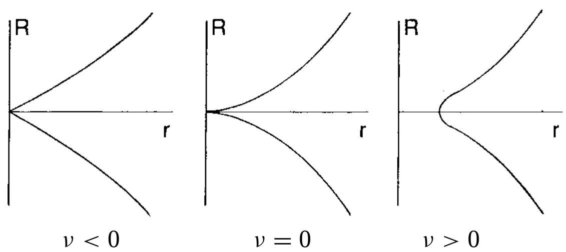
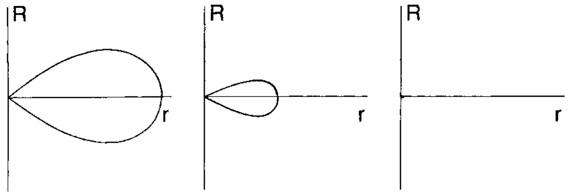


Figure 1. Case A: $\eta < 0$.



$\nu \ll 0$ $\nu < 0$ $\nu = 0$

Figure 2. Case B: $\eta > 0$.

In the Case A when $\eta < 0$ the stable and unstable manifolds detach from the equilibrium as a single invariant manifold as ν increases from zero.

In the Case B when $\eta > 0$ the stable and unstable manifolds are globally connected for $\nu < 0$ and shrink to the equilibrium as $\nu \to 0^-$.

These statements hold for the truncated system with Hamiltonian (4) only, but they are a good first approximation of the local evolution of the stable and unstable manifolds. A complete analytic and tedious analysis of the full system with higher order terms will be forthcoming in [10].

4. Strömgren's Conjecture and the Blue Sky Catastrophe of Devaney

Strömgren conjectured based on numerical evidence that there were orbits doubly asymptotic to $\mathcal{L}_4$ in the restricted three-body problem and that these doubly asymptotic orbits are the limit of periodic orbits with long periods (the blue sky catastrophe). Henrard [6] and Devaney [4] established general theorems which would verify Strömgren's conjecture provided the stable and unstable manifolds at $\mathcal{L}_4$ intersect transversally in the $H =$ constant level set.

The Hamiltonian of the restricted problem at $\mathcal{L}_4$ can be considered as a perturbation of the Hamiltonian (3) in the Case B when $\eta > 0$. Thus, to the first approximation when $\mu > \mu_1$ and $\mu \sim \mu_1$ the stable and unstable manifolds are globally connected. Using symplectic manifold intersection theory one can show that the stable and unstable manifolds at $\mathcal{L}_4$ intersect for $\mu > \mu_1$ [10]. Of course, a normal form argument will never show a transversal intersection!

5. A Correction

The quantity ν for the restricted problem has been computed by various people in various forms. To my knowledge the first calculation was done in 1968 by Deprit and Henrard [3] to complete the 1941 theorem of Buchanan [2]. To show that η was positive in the restricted problem Meyer and Schmidt used this calculation in their 1971 paper on the Hamiltonian–Hopf bifurcation [12]. The formula for g (essentially

η) on page 107 of [12] should not contain the $\sqrt{2}$. This calculation shows that $\eta > 0$ also.

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References

1. Arnold, V. I.: Proof of A. N. Kolmogorov's theorem on the preservation of quasiperiodic motions under small perturbations of the Hamiltonian, *Russ. Math. Surv.* **18**(5) (1963), 9–36.
2. Buchanan, D.: Trojan satellites – limiting case, *Trans. R. Soc. Canada* **35**, 9–25.
3. Deprit, A. and Henrard, J.: A manifold of periodic solutions, *Adv. Astron. Astrophy.* **6** (1968), 6–124.
4. Devaney, R. L.: Blue sky catastrophes in reversible and Hamiltonian systems, *Ind. Univ. Math. J.* **26**(2) (1977), 247–263.
5. Hale, J. K.: *Ordinary Differential Equations*, Wiley, New York, 1972.
6. Henrard, J.: Proof of a conjecture of E. Strömgren, *Celest. Mech.* **7** (1973), 449–457.
7. Laub, A. and Meyer, K. R.: Canonical forms for symplectic and Hamiltonian matrices, *Celest. Mech.* **9** (1974), 213–238.
8. Golubitsky, M., Marsden, J., Stewart, I. and Dellnitz, M.: The constrained Liapunov–Schmidt procedure and periodic orbits, *Normal Forms and Homoclinic Chaos*, Fields Institute Communications 4, Amer. Math. Soc., Providence, RI, 1991.
9. Lypounoff, A.: Problem général de la stablité du mouvement, *Ann. Math. Stud.*, 17, Princeton Univ. Press, Princeton, NJ, 1947.
10. McSwiggen, P. and Meyer, K. R.: The evolution of invariant manifolds, manuscript.
11. Meyer, K. R. and Hall, G. R.: *Introduction to Hamiltonian Dynamical Systems and the N-body Problem*, Springer-Verlag, New York, 1992.
12. Meyer, K. R. and Schmidt, D. S.: Periodic orbits near L_4 for mass ratios near the critical mass ratio of Routh, *Celest. Mech.* **4** (1971), 99–109.
13. Schmidt, D. S.: Versal normal form of the Hamiltonian function of the restricted problem of three bodies near $\mathcal{L}_4$, *J. Comp. and Appl. Math.* **52** (1994), 155–176.
14. Sokol'skii, A.: On the stability of an autonomous Hamiltonian system, *J. Appl. Math. Mech.* **38** (1978), 741–749.
15. Williams, J.: On the normal forms of linear canonical matrices, *Amer. J. Math.* **59** (1937), 599–617.