LIBRATIONS OF CENTRAL CONFIGURATIONS AND BRAIDED SATURN RINGS

KENNETH R. MEYER
Institute for Dynamics, Department of Mathematics, University of Cincinnati, Cincinnati, OH 45221-0025, U.S.A.

and

DIETER S. SCHMIDT
Institute for Dynamics, Department of Computer Science, University of Cincinnati, Cincinnati, OH 45221-0008, U.S.A.

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Abstract. We give a simple mathematical model for braided rings of a planet based on Maxwell's model for the rings of Saturn.

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1. Introduction

Highly symmetric central configurations are useful models for planetary rings. Indeed, the first stable model of a single planetary ring is the central configuration of the $(N + 1)$-body problem where one massive body is at the origin and N small bodies of equal mass are at the vertices of a regular N-agon. The proof of the stability of this central configuration appears in the prize winning essay *On the Stability of the Motion of Saturn's Rings* by Maxwell (1859) – also see Moeckel (1992). The N small bodies might be following a circular orbit, in which case the ring would look like a stationary circle, or they might follow an elliptic orbit, in which case the ring would look a pulsating circle. In either case the configuration is planar.

The pictures of the rings of Saturn returned by the fly-by of Voyager 1 suggested that the F-ring consisted of three components. The two outer components appeared to be intertwined or braided. See Smith *et al.* (1981). To better understand this phenomena, several regions of the F-ring were selected for repeated and high resolution observation during the fly-by of Voyager 2. "Voyager 2 found the same clumpy and occasionally kinked appearance of the F ring, but surprisingly found only one small region where the rings appear twisted or braided" – Smith *et al.* (1982). The words 'clumpy', 'kinked', 'twisted', and 'braided' suggest nonplanar orbits. This suggestion was the motivation for our research into nonplanar periodic solutions on the N-body problem which might be considered as twisted rings even though Smith *et al.* (1982) and subsequent researchers believe that the actual rings of Saturn are essentially planar.

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For dense planetary rings the collision of individual particles have to be taken into account. The F-ring appears to be rather diffuse so that in the simplified models we consider we will assume that no collision between the particles will occur.

Lissauer and Peale (1986) made the further simplifying assumption that there will be no gravitational interaction between the particles in the ring. On the other hand they include the effect of a small *shepherding moon*, that is, they start with a planar circular restricted three-body problem (Saturn, the shepherding moon and one particle in the ring). They then consider N equally spaced ring particles and they follow their motion via numerical integration. When suitable initial conditions are chosen the particles follow a path which is knows as *Brown's horseshoe*, when it is viewed in a coordinate system which rotates with the moon. The doubling back of the particles on the horseshoe is their explanation for the braided appearance of the F-ring. Their model is planar.

Our model does not take the effect of a shepherding moon into account but we do include the gravitational interaction of the individual particles. We assume that the particles maintain a high degree of symmetry all times and that they are close to a central configuration.

Central configurations by necessity must be planar – Wintner (1941). However, a central configuration can be considered as an equilibrium point of the equations of motion in a rotating three dimensional coordinate system. We use standard linearization and normalization methods to establish the existence of three dimensional periodic solutions near this equilibrium solution.

Davies, Truman and Williams (1983, in prep) numerically investigated several highly symmetric three dimensional subsystems of the N-body problem. They consider systems with both gravitational and Coulomb forces and they found a wealth of three dimensional periodic solutions. We consider systems very similar to theirs with the goal of finding periodic solutions. However, our models are for three dimensional planetary rings.

It is popular today to use group theoretic ideas to discuss mechanical systems with symmetries even though the use of these methods in mechanics goes back to the last century. For continuous groups of symmetries there is the classical Noether theorem on the existence of integrals and the reduction theorem of Meyer (1973), Marsden and Weinstein (1974). For reflective symmetries such as the symmetry in the line of syzygy in the restricted three body problem see Meyer (1981). The problems discussed by Davis *et al.* (1983, in prep) and in this paper admit a finite group of symplectic symmetries. A theorem found in Guillemin and Sternberg (1984) states that space fixed by all the elements of such a finite group of symmetries is an invariant symplectic subspace. In general, if the group is large the fixed space is small. We use this theorem to give a simple and rigorous derivation of our three dimensional ring models.

Our rings models are Hamiltonian systems of two degrees of freedom which have an equilibrium point which corresponds to a classical central configuration. By linearizing about this equilibrium and applying Liapunov's center theorem we obtain periodic solutions which might be called 'kinked' rings. By normalizing the Hamiltonian through the fourth order and applying Birkhoff's theorem or KAM theory we obtain periodic solutions which might be called 'twisted' or 'braided'.

2. The Nonalternating $(N+1)$-Body Problem

Consider the $(N+1)$-body problem with the center of mass fixed at the origin in a rotating coordinate system in $\mathbf{R}^3$ (rotating about the z-axis). Let $\mathbf{q}_0, \mathbf{q}_1, \ldots, \mathbf{q}_N$ be the position vectors, $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_N$ be the momentum vectors of $N+1$ particles of masses $m_0 = M$, $m_1 = \ldots = m_N = \varepsilon = 1/N$.

We think of M as large (all though this is completely unnecessary from a mathematical point of view) and so we will refer to the zeroth particle as the massive body or the planet. Since $\mathbf{q}_i, \mathbf{p}_i \in \mathbf{R}^3$ for $i = 0, \ldots, N$ the phase space, $\mathbf{V}_N$, is the symplectic subspace of $\mathbf{R}^{6N+6}$ where $M\,\mathbf{q}_0 + \varepsilon\,\mathbf{q}_1 + \cdots + \varepsilon\,\mathbf{q}_N = 0$ (the center of mass is at the origin), and $\mathbf{p}_0 + \mathbf{p}_1 + \cdots + \mathbf{p}_N = 0$ (linear momentum is zero). The Hamiltonian of this problem is

$$H_N = \frac{\|\mathbf{p}\|^2}{2M} - \mathbf{q}_0^T\,\mathbf{L}\,\mathbf{p}_0 + \sum_{i=1}^{N}\left(\frac{\|\mathbf{p}\|^2}{2\varepsilon} - \mathbf{q}_i^T\,\mathbf{L}\,\mathbf{p}_i\right) +$$

$$- \sum_{1\leq i<j\leq N} \frac{\varepsilon^2}{\|\mathbf{q}_i - \mathbf{q}_j\|} - \sum_{i=1}^{N} \frac{\varepsilon\, M}{\|\mathbf{q}_i - \mathbf{q}_0\|},$$

where

$$L = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The $(N+1)$-body problem admits many symmetries: translational, rotational and sometimes finite symmetries. Since we have fixed the center of mass at the origin we have made the reduction due to the translational symmetry in the classical manner. Some of the rotational symmetry is fogged due to the fact that the problem is written in rotating coordinates, but it is clear that the problem is still invariant under rotations about the z-axis. Thus the problem admits the component of angular momentum in the z-direction as an integral and the problem can be further reduced by considering two configurations equivalent if they differ by a rotation. This reduction is done in the classical manner by introducing polar coordinates in the x, y plane, holding angular momentum fixed, and ignoring the polar angle. This reduction will be done subsequently. For now consider an additional symmetry in the problem that arises due to the fact that N of the bodies are of equal mass.

Consider the group $\mathcal{G}_N$ of symmetries generated by the transformations

$$R_0 = \begin{cases} (\mathbf{q}_0, \mathbf{p}_0) \to (\mathbf{A}\,\mathbf{q}_0, \mathbf{A}\,\mathbf{p}_0) \\ (\mathbf{q}_i, \mathbf{p}_i) \to (\mathbf{q}_i, \mathbf{p}_i) \qquad \text{for } i = 1, ..., N \end{cases}$$

$$R_1 = \begin{cases} (\mathbf{q}_0, \mathbf{p}_0) \to (\mathbf{q}_0, \mathbf{p}_0) \\ (\mathbf{q}_i, \mathbf{p}_i) \to (\mathbf{A}\,\mathbf{q}_{i-1}, \mathbf{A}\,\mathbf{p}_{i-1}) \quad \text{for } i = 2, ..., N \\ (\mathbf{q}_1, \mathbf{p}_1) \to (\mathbf{A}\,\mathbf{q}_N, \mathbf{A}\,\mathbf{p}_N) \end{cases}$$

where

$$A = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \varphi = 2\pi/N\,.$$

These two transformations generate a finite group of symmetries since $\mathbf{A}^N = \mathbf{I}$. Each of these transformations is linear and symplectic. They leave the center of mass fixed at the origin and total linear momentum fixed at zero. Thus these transformations generate a finite group of linear, symplectic transformations of the linear symplectic space $\mathbf{V}_N$. The Hamiltonian, H_N, is also invariant under these transformations. The following theorem is a corollary of a more general theorem found in Guillemin and Sternberg (1984), p. 203.

THEOREM. *Let $\mathcal{G}$ be a finite group of linear symplectic transformations of a symplectic linear space* $\mathbf{V}$. *Then the fixed set,* $\mathbf{F} = \{v \in \mathbf{V} \;:\; gv = v \text{ for all } g \in \mathcal{G}\}$, *is a symplectic subspace of* $\mathbf{V}$. *Moreover, if H is a Hamiltonian invariant under each of the transformations of $\mathcal{G}$, then* $\mathbf{F}$ *is an invariant subspace for the Hamiltonian flow defined by H.*

Let $\mathbf{q}_i = (x_i, y_i, z_i)^T$ and $\mathbf{p}_i = (X_i, Y_i, Z_i)^T$. A point is fixed under the transformation R_0 if $\mathbf{q}_0 = (0, 0, z_0)^T$ and $\mathbf{p}_0 = (0, 0, Z_0)^T$ and it is fixed under R_1 if $\mathbf{q}_i = \mathbf{A}\,\mathbf{q}_{i-1}$, $\mathbf{p}_i = \mathbf{A}\,\mathbf{p}_{i-1}$ for $i = 2, ..., N$ and $\mathbf{q}_1 = \mathbf{A}\,\mathbf{q}_N$, $\mathbf{p}_1 = \mathbf{A}\,\mathbf{p}_N$. In particular, $z_1 = z_2 = \cdots = z_N$ and $Z_1 = Z_2 = \cdots = Z_N$. Since the center of mass is at the origin and total linear momentum is zero $M\,z_0 + \varepsilon z_1 + \cdots + \varepsilon z_N = 0$ and $Z_0 + Z_1 + \cdots + Z_N = 0$. Thus the fixed set, $\mathbf{F}_N$, for this group is the set of configurations where the planet remains on the z-axis and the N particles lie at the vertices of a regular N-agon which lies in a plane parallel to the x, y plane. See Figure 1.

Therefore $(x, y, z) = (x_1, y_1, z_1)$ and $(X, Y, Z) = (X_1, Y_1, Z_1)$ are coordinates on $\mathbf{F}_N$ and the Hamiltonian restricted to this invariant set is

$$S_N = \frac{N^2 Z^2}{2M} + \frac{N}{2\varepsilon}\,(X^2 + Y^2 + Z^2) - N(xY - yX) +$$

$$- \frac{C_N}{(x^2+y^2)^{1/2}} - \frac{M}{(x^2+y^2+(1+1/M)^2 z^2)^{1/2}} ,$$

where C_N is the constant

$$C_N = \frac{(x^2+y^2)^{1/2}}{N^2} \sum_{1 \le i < j \le N} \frac{1}{\|\mathbf{q}_i - \mathbf{q}_j\|} =$$

$$= \frac{1}{2N} \sum_{j=1}^{N-1} \frac{1}{|1-\omega^j|} = \frac{1}{4N} \sum_{j=1}^{N-1} \frac{1}{\sin(\pi j/N)}$$

and ω is the Nth root of unity. Since $\sin(\pi j/N) \ge 2j/N$ for $j = 1, ..., N/2$, it follows that $C_n \to +\infty$ as $N \to +\infty$.

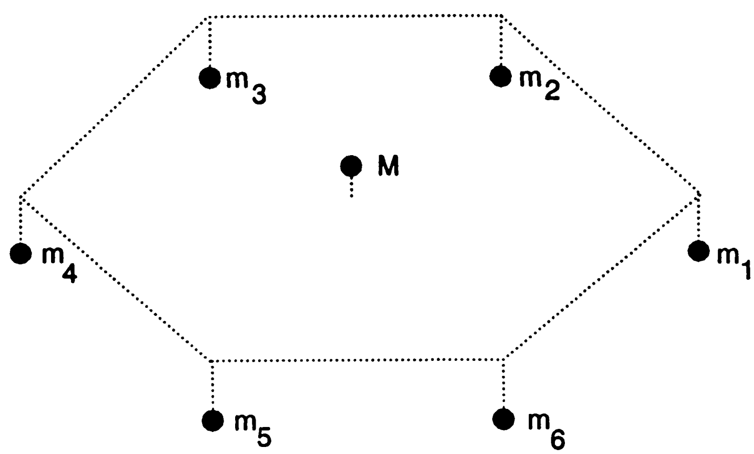

Fig. 1. The nonalternating $(N+1)$-configuration.

The Hamiltonian S_N is invariant under a rotation about the z-axis and so can be reduced by one more degree of freedom, Meyer (1973). To make this reduction change to symplectic polar coordinates $(r, \vartheta, R, \Theta)$ by

$$x = r \cos \vartheta , \quad X = R \cos \vartheta - (\Theta/r) \sin \vartheta ,$$

$$y = r \sin \vartheta , \quad Y = R \sin \vartheta + (\Theta/r) \cos \vartheta ,$$

to get

$$S_N = (1+1/M)N^2 \frac{Z^2}{2} + \frac{N^2}{2} + \left(R^2 + \frac{\Theta^2}{r^2}\right) - N\Theta +$$

$$- \frac{C_N}{r} - \frac{M}{\{r^2 + (1+1/M)^2 z^2\}^{1/2}} .$$

Scale by $Z \to Z/N, R \to R/N, \Theta \to \Theta/N$ (this is a symplectic transformation with multiplier N), and scale time by $t \to Nt$, so that altogether $K_N = S_N$. Since the Hamiltonian is independent of ϑ (ϑ is ignorable) the angular momentum Θ is an integral, so let $\Theta = \alpha$ where α is constant. The Hamiltonian is then

$$K_N = (1+1/M)\,\frac{Z^2}{2} + \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \Theta + \\ - \frac{C_N}{r} - \frac{M}{\{r^2 + (1+1/M)^2 z^2\}^{1/2}}\,.$$

The equations of motion are

$$\dot{r} = R\,, \qquad \dot{R} = \frac{\alpha^2}{r^3} - \frac{C_N}{r^2} - \frac{Mr}{\{r^2 + (1+1/M)^2 z^2\}^{3/2}}\,,$$

$$\dot{z} = (1+1/M)Z\,, \qquad \dot{Z} = \frac{-M(1+1/M)^2 z}{\{r^2 + (1+1/M)^2 z^2\}^{3/2}}\,.$$

3. The Alternating $(2N+1)$-Body Problem

Consider the $(2N+1)$-body problem with center of mass fixed at the origin in a rotating coordinate system in $\mathbf{R}^3$ rotating about the z-axis. Let $\mathbf{q}_0, \mathbf{q}_1, \ldots, \mathbf{q}_{2N}$ be the position vectors, $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_{2N}$ be the momentum vectors of the $2N+1$ particles of masses $m_0 = M$, $m_1 = \cdots = m_{2N} = 1/(2N) = \varepsilon$.

Again we will refer to the zeroth particle as the massive body or the planet. Since $\mathbf{q}_i, \mathbf{p}_i \in \mathbf{R}^3$ for $i = 0, \ldots, 2N$ the phase space is the symplectic subspace $\mathbf{V}_{2N}$ of $\mathbf{R}^{12N+6}$ where $M\,\mathbf{q}_0 + \varepsilon\,\mathbf{q}_1 + \cdots + \varepsilon\,\mathbf{q}_{2N} = 0$ and $\mathbf{p}_0 + \mathbf{p}_1 + \cdots + \mathbf{p}_{2N} = 0$. The Hamiltonian of this problem is

$$H_{2N} = \frac{\|\mathbf{p}\|^2}{2M} - \mathbf{q}_0^T\,\mathbf{L}\,\mathbf{p}_0 + \sum_{i=1}^{2N}\left(\frac{\|\mathbf{p}\|^2}{2\varepsilon} - \mathbf{q}_i^T\,\mathbf{L}\,\mathbf{p}_i\right) + \\ - \sum_{1\le i<j\le 2N} \frac{\varepsilon^2}{\|\mathbf{q}_i - \mathbf{q}_j\|} - \sum_{i=1}^{2N} \frac{\varepsilon\, M}{\|\mathbf{q}_i - \mathbf{q}_0\|}\,,$$

where L is as above.

Consider the group $\mathcal{G}_{2N}$ of symmetries generated by the transformations

$$R_0 = \begin{cases} (\mathbf{q}_0, \mathbf{p}_0) \to (\mathbf{A}\,\mathbf{q}_0, \mathbf{A}\,\mathbf{p}_0) & \\ (\mathbf{q}_i, \mathbf{p}_i) \to (\mathbf{q}_i, \mathbf{p}_i) & \text{for } i = 1, \ldots, 2N \end{cases}$$

$$R_1 = \begin{cases} (\mathbf{q}_0, \mathbf{p}_0) \to (\mathbf{q}_0, \mathbf{p}_0) \\ (\mathbf{q}_i, \mathbf{p}_i) \to (\mathbf{A}\,\mathbf{q}_{i-1}, \mathbf{A}\,\mathbf{p}_{i-1}) & \text{for } i = 2, ..., 2N \\ (\mathbf{q}_1, \mathbf{p}_1) \to (\mathbf{A}\,q_{2N}, \mathbf{A}\,\mathbf{p}_{2N}) \end{cases}$$

where

$$A = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and now} \quad \varphi = \pi/N\,.$$

Note the negative sign in the 3,3 position of **A** in this example! These two transformations generate a finite group of symmetries since $\mathbf{A}^{2N} = \mathbf{I}$. Each of these transformations is linear and symplectic. They leave the center of mass fixed at the origin and total linear momentum fixed at zero. Thus, these transformations generate a finite group of linear, symplectic transformations of the linear symplectic space $\mathbf{V}_{2N}$. The Hamiltonian, H_{2N}, is also invariant under these transformations.

Let $\mathbf{q}_i = (x_i, y_i, z_i)^T$ and $\mathbf{p}_i = (X_i, Y_i, Z_i)^T$. A point is fixed under the transformations R_0 if $\mathbf{q}_0 = (0, 0, z_0)^T$ and $\mathbf{p}_0 = (0, 0, Z_0)^T$ and is fixed under R_1 if $\mathbf{q}_i = \mathbf{A}\,\mathbf{q}_{i-1}$, $\mathbf{p}_i = \mathbf{A}\,\mathbf{p}_{i-1}$ for $i = 2, ..., 2N$ and $\mathbf{q}_1 = \mathbf{A}\,\mathbf{q}_{2N}$, $\mathbf{p}_1 = \mathbf{A}\,\mathbf{p}_{2N}$. Note that now $z_{i+1} = -z_i$ and $Z_{i+1} = -Z_i$ for $i = 1, ..., 2N - 1$. The small particles alternate up and down around the ring. Since the center of mass is at the origin and linear momentum is zero it follows that $z_0 = Z_0 = 0$. Thus a configuration is in the fixed set, $\mathbf{F}_{2N}$, if the planet is fixed at the origin and the other $2N$ particles are alternatively above and below the vertices of a regular $2N$-agon in the x, y plane (see Figure 2).

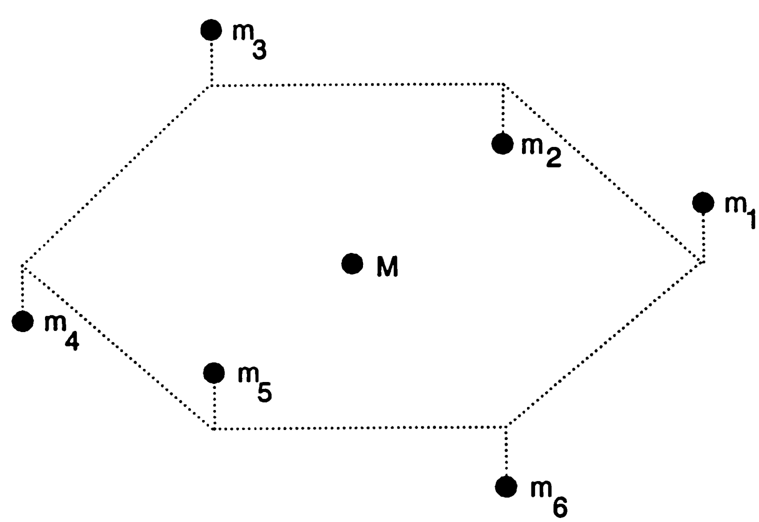

Fig. 2. The alternating $(2N + 1)$-configuration.

Proceed as in the example above. On $\mathbf{F}_{2N}$, the fixed set, $\mathbf{q}_0 = \mathbf{p}_0 = 0$,

$\mathbf{q}_j = \mathbf{A}^{j-1}\,\mathbf{q}_1, \mathbf{p}_j = \mathbf{A}^{j-1}\,\mathbf{p}_1$ for $j = 1, ..., 2N$. The center of mass is at the origin, $M\,\mathbf{q}_0 + \varepsilon\,\mathbf{q}_1 + \cdots + \varepsilon\,\mathbf{q}_{2N} = 0$, and linear momentum is zero, $\mathbf{p}_0 + \mathbf{p}_1 + \cdots + \mathbf{p}_{2N} = 0$. Therefore $\mathbf{u} = \mathbf{q}_1 = (x, y, z)$ and $\mathbf{U} = \mathbf{p}_1 = (X, Y, Z)$ are coordinates on $\mathbf{F}_{2N}$ and the Hamiltonian restricted to this invariant set is

$$S_{2N} = 2N\,\frac{(X^2 + Y^2 + Z^2)}{2\varepsilon} - 2N(xY - yX) + \\ - \sum \frac{\varepsilon^2}{\|A^{i-1}\mathbf{u} - A^{j-1}\mathbf{u}\|} - \frac{2N\,\varepsilon\,M}{(x^2 + y^2 + z^2)^{1/2}}\,.$$

The third term can be written differently. Let $r^2 = x^2 + y^2$ and ω be the $(2N)$th root of unity.

$$\sum_{i<j} \frac{\varepsilon^2}{\|\mathbf{A}^{i-1}\mathbf{u} - \mathbf{A}^{j-1}\,\mathbf{u}\|} =$$

$$= \frac{1}{4N} \sum_{k=1}^{2N-1} \frac{1}{\{|1 - \omega^k|^2 r^2 + (1 - (-1)^k)^2 z^2\}^{1/2}} =$$

$$= \frac{1}{8N} \sum_{k=1}^{2N-1} \frac{1}{\{a_k^2 r^2 + b_k z^2\}^{1/2}}$$

where $a_k = \sin(\pi k/2N)$ and $b_k = k \bmod 2$, i.e. b_k is 0 for even k and 1 when k is odd. Change to polar coordinates $(r, \vartheta, R, \Theta)$ as before to get

$$S_{2N} = \frac{(2N)^2}{2}\left(R^2 + \frac{\Theta^2}{r^2} + Z^2\right) - 2N\Theta + \\ - \frac{1}{8N} \sum_{k=1}^{2N-1} \frac{1}{\{a_k^2 r^2 + b_k z^2\}^{1/2}} - \frac{M}{(r^2 + z^2)^{1/2}}\,.$$

Since the Hamiltonian is independent of ϑ the momenum, Θ, is an integral, so let $\Theta = \alpha$. Scale by $Z \to Z/2N$, $R \to R/2N$, $\Theta \to \Theta/2N$ (symplectic with multiplier $2N$), and scale time by $t \to 2Nt$, so that in the end $K_{2N} = S_{2N}$.

$$K_{2N} = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2} + Z^2\right) - \Theta + \\ - \frac{1}{8N} \sum_{k=1}^{2N-1} \frac{1}{\{a_k^2 r^2 + b_k z^2\}^{1/2}} - \frac{M}{(r^2 + z^2)^{1/2}}\,.$$

The equations of motion are

$$\dot{r} = R\,, \qquad \dot{R} = \frac{\alpha^2}{r^3} - \frac{1}{8N}\sum_{k=1}^{2N-1} \frac{a_k^2 r}{\{a_k^2 r^2 + b_k z^2\}^{3/2}} - \frac{Mr}{(r^2+z^2)^{3/2}}$$

$$\dot{z} = Z\,, \qquad \dot{Z} = -\frac{1}{8N}\sum_{k=1}^{2N-1} \frac{b_k z}{\{a_k^2 r^2 + b_k z^2\}^{3/2}} - \frac{Mz}{(r^2+z^2)^{3/2}}\,.$$

4. Linearized Nonalternating $(N+1)$-Body Problem

By setting $\alpha^2 = C_N + M$ there is an equilibrium point at $R = Z = z = 0$, $r = 1$. The linearized equations near this point with $r = 1 + \rho$ are

$$\dot{\rho} = R\,, \quad \dot{R} = -\alpha^2\rho\,, \quad \dot{z} = (1 + 1/M)Z\,, \quad \dot{Z} = -M(1+1/M)^2 z\,.$$

These equations come from the quadratic terms of the Hamiltonian, namely

$$P_N = \frac{1}{2}\left(\alpha^2\rho^2 + M(1+1/M)^2 z^2 + R^2 + (1+1/M)Z^2\right).$$

For all positive values of the parameter M these are the equations of two harmonic oscillators with frequencies α and ω where $\omega^2 = (1+M)^3/M^2$. Except when the ratio of the frequencies α/ω is an integer or the reciprocal of an integer Liapunov's center theorem gives that there are two one-parameter families of periodic solutions emanating from the equilibrium (see Meyer and Hall, 1991). At the equilibrium one family is tangent to the plane $z = Z = 0$ (we will call this the z-mode) and the other family is tangent to $r = 1$, $R = 0$ (we will vall this the r-mode). The Hamiltonian P_N is positive definite so a theorem of Weinstein (1973) assures us that there are two periodic solutions near the equilibrium in each level set of the Hamiltonian K_N even when the ratio of the frequencies α/ω is an integer or the reciprocal of an integer.

The $z = Z = 0$ plane is the original x, y plane and it is clearly invariant and the r-mode solutions lie in this invariant plane. The periodic solutions in this plane have frequency approximately equal to $\alpha - 1$. Recall that the ignored polar angle ϑ has the equation $\dot{\vartheta} = \alpha - 1$. So in the x, y plane when the polar angle increases by 2π the radial variable r undergoes one cycle. This is an elliptic orbit in the plane! Recall that the equilibrium point corresponds to a central configuration and there are solutions of the $(N+1)$-body problem where the bodies move on a Kepler orbit (for example an elliptic orbit) while remaining in a central configuration (see Meyer and Hall, 1991). Thus, this family gives no new information.

The plane $r = 1$, $R = 0$ corresponds to a cylinder in x, y, z space and is definitely not planar and the family of z-mode solutions are tangent to this cylinder. Going back to the original coordinates this family of periodic solution gives rise to a solution of the $(N+1)$-body problem which looks like the massive planet surrounded by a ring of small particles and the ring remains planar but oscillates

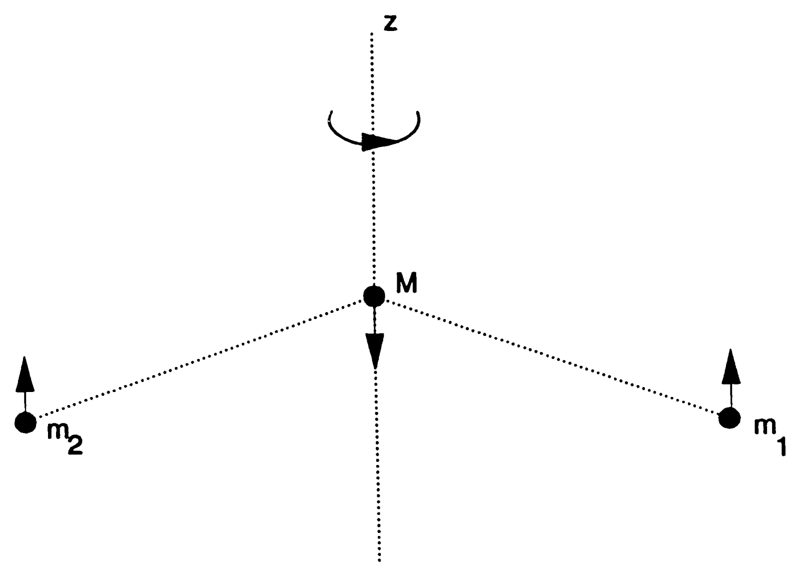

Fig. 3. An oscillating ring.

up and down. See Figure 3. In this and subsequent figures the two dimensional drawing is in a rotating plane which rotates about the z-axis. Thus the particles as of mass m_i oscillate up and down in the plane as the plane rotates. Therefore the particles sweep out a circle in three space which oscillates up and down.

5. Linearized Alternating $(2N + 1)$-Body Problem

By setting $\alpha^2 = C_{2N} + M$ there is one equilibrium point at $R = Z = z = 0$, $r = 1$. The linearized equations about this equilibrium point are

$$\dot{\rho} = R\,, \quad \dot{E} = -\alpha^2 \rho\,, \quad \dot{z} = Z\,, \quad \dot{Z} = -(D_{2N} + M)z\,,$$

where $\rho = 1 - r$ and

$$D_{2N} = \frac{1}{8N} \sum_{k=1}^{2N-1} \frac{b_k}{a_k} = \frac{1}{8N} \sum_{j=1}^{N} \sin^{-1} \frac{\pi(2j-1)}{2N} = C_{2N} - \frac{1}{2} C_N\,.$$

These equations follow from the quadratic Hamiltonian

$$2P_{2N} = \alpha^2 \rho^2 + (D_{2N} + M)z^2 + R^2 + Z^2\,.$$

For all values of the parameter M these are the equations of two harmonic oscillators with frequencies α and ω where $\omega^2 = D_{2N} + M$. Except when the ratio of the frequencies α/ω is an integer or the reciprocal of an integer Liapunov's center theorem gives that there are two one-parameter families of periodic solutions

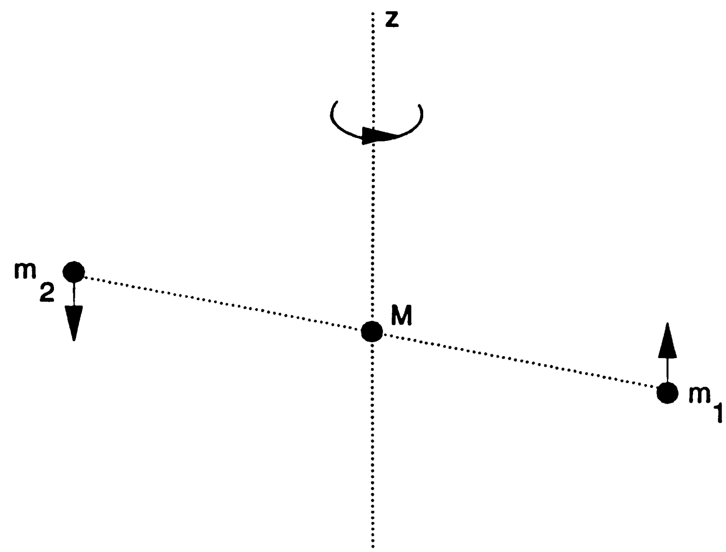

Fig. 4. A kinked ring.

emanating from the equilibrium (see Meyer and Hall, 1991). These families are tangent at the equilibrium to the two planes $r = 1$, $R = 0$ (the r-mode) and $z = Z = 0$ (the z-mode). The Hamiltonian P_{2N} is positive definite so a theorem of Weinstein (1973) assures us that there are two periodic solutions in each level set of the Hamiltonian K_N even when the ratio of the frequencies α/ω is an integer or the reciprocal of an integer.

The $z = Z = 0$ plane is the original x, y plane and it is clearly invariant. As in the example above this family comes from the elliptic solutions and this family gives no new information.

The plane $r = 1$, $R = 0$ corresponds to a cylinder in x, y, z space and is definitely not planar and the family of z-mode solutions is tangent to this cylinder. Going back to the original coordinates this family of periodic solution gives rise to a solution of the $(2N + 1)$-body problem which looks like a fixed massive planet surrounded by a ring of small particles and the ring is not planar. The particles in the ring oscillate up and down with each one out of phase with its two neighbors. This solution might be called 'kinked'. See Figure 4.

6. Normalized Nonalternating $(N + 1)$ and Alternating $(2N + 1)$ Problems

In order to study the given problems near the equilibrium points we expand the Hamiltonian into a sum of homogeneous polynomials. Due to the special nature of the problems, these polynomials will contain only the variables ρ and z, and in addition to this the exponents of z are always even. The Hamiltonian for both cases through fourth order terms has the form

$$K = K_2 + a_0\rho^3 + a_1\rho z^2 + b_0\rho^4 + b_1\rho^2 z^2 + b_2 z^4 + \cdots$$

with

$$K_2 = \frac{1}{2}\left(\alpha^2\rho^2 + \gamma z^2 + R^2 + \beta Z^2\right)$$

representing the quadratic terms in either case.

The first step in normalizing a Hamiltonian function near an equilibrium point is always to bring these quadratic terms into a canonical form. In the two problems under discussion the linearized systems are already close to the real canonical form as they are

$$\dot\rho = R\,,\quad \dot R = -\alpha^2\rho\,,\quad \dot z = \beta Z\,,\quad \dot Z = -\gamma z\,.$$

This allows at once the introduction of canonical action angle variables $(I_1, I_2, \varphi_1, \varphi_2)$ via the transformation

$$\rho = \sqrt{2I/\alpha}\,\cos\varphi_1\,, \qquad R = \sqrt{2I\alpha}\,\sin\varphi_1\,,$$

$$z = \sqrt{2I_2\sqrt{\beta/\gamma}}\,\cos\varphi_2\,, \quad Z = \sqrt{2I_2\sqrt{\beta/\gamma}}\,\sin\varphi_2\,,$$

where $\omega_1 = \alpha$ and $\omega_2 = \sqrt{\gamma\beta}$ are the frequencies of the linearized system. With the help of these frequencies the above transformation can be written in a more symmetric form, that is as

$$r = \sqrt{2I_1/\omega_1}\,\cos\varphi_1\,, \qquad R = \sqrt{2I_1\omega_1}\,\sin\varphi_1\,,$$

$$z = \sqrt{2I_2\beta/\omega_2}\,\cos\varphi_2\,, \quad Z = \sqrt{2I_2\omega_2/\beta}\,\sin\varphi_2\,.$$

For the nonalternating $N+1$ bodies the two frequencies are

$$\omega_1^2 = M + C_N\,, \qquad \omega_2^2 = (M+1)^3/M^2$$

and for the alternating $2N+1$ bodies they are

$$\omega_1^2 = M + C_{2N}\,, \qquad \omega_2^2 = M + D_{2N}\,.$$

The next step is the transformation of the full Hamiltonian function to the action angle variables. This leads to the Hamiltonian

$$H = \omega_1 I_1 + \omega_2 I_2 + \sum_{k>2} H_k(\sqrt{I_1}, \sqrt{I_2}, \varphi_1, \varphi_2)$$

with H_k Poisson series of degree $k/2$ in the action variables. Due to the special form of the given problems there are restrictions on the linear combinations of angles which can appear within each H_k. Because of this only a few resonance conditions between ω_1 and ω_2 have to be excluded when we put the Hamiltonian into normal form.

We used the method of Lie transformation of Deprit (1969) and checked our calculations with the help of MACSYMA. The normal form of the above Hamiltonian through fourth order terms is

$$H = \omega_1 I_1 + \omega_2 I_2 + H_2(I_1, I_2) + O_5$$

provided that the following two resonance conditions are excluded: $\omega_1 = \omega_2$ and $\omega_1 = 2\omega_2$. The quadratic terms are written as

$$H_2 = \frac{1}{2}\,\{AI_1^2/\omega_1^2 + 2BI_1I_2\beta/(\omega_1\omega_2) + CI_2^2\beta^2/\omega_2^2\}$$

with the coefficients

$$A = 3b_0 - 15a_0^2/(2\omega_1^2)$$

$$B = b_1 - 3a_0a_1\omega_2^2/\omega_1^2 - 2a_1^2\beta/(4\omega_2^2 - \omega_1^2)$$

$$C = 3b_2 + a_1^2(3\omega_1^2 - 8\omega_2^2)/(2\omega_1^2(4\omega_2^2 - \omega_1^2))\,.$$

In order to apply the KAM theory we need to calculate that quantity the

$$D = H_2(\omega_2, -\omega_1)$$

is nonzero. The quantity D for the K given above is

$$D = \frac{-15a_0^2\omega_2^2}{4\omega_1^4} + \frac{3\beta a_0 a_1}{\omega_1^2} + \frac{3\beta^2 a_1^2\omega_1^2}{8\omega_2^2(4\omega_2^2 - \omega_1^2)} + \frac{3b_0\omega_2^2}{2\omega_1^2} - \beta b_1 + \frac{3\beta^2 b_2\omega_1^2}{2\omega_2^2}\,.$$

The nonalternating $(N+1)$-bodies problem depends on the two parameters, the mass M of the central body and the number N of infinitesimal bodies. Since the frequencies of the linearized system are $\omega_1^2 = M + C_N$ and $\omega_2^2 = (M+1)^3/M^2$ the last expression can be written in a variety of different ways. One of them is

$$D_N = \frac{9}{16}\,\omega_1^2\omega_2^2\left(\frac{3}{4\omega_2^2 - \omega_1^2} - \frac{1}{M}\right).$$

In a similar fashion the frequencies for the alternating $2N+1$ bodies are $\omega_1^2 = M + C_{2N}$ and $\omega_2^2 = M + D_{2N}$. But now the expansion of the Hamiltonian function into a Taylor series has to be carried out inside the summation over the $2N$ bodies. This introduces additional expressions which depend on N. With the

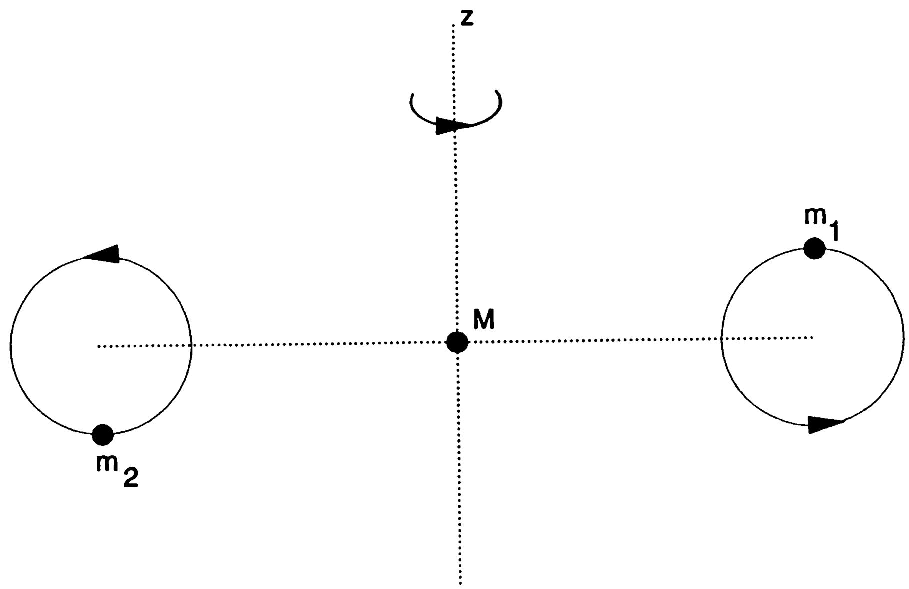

Fig. 5. A braided ring.

notation

$$E_{2N} = \frac{1}{8N} \sum_{k=1}^{2N-1} \frac{b_k}{a_k^5} = \frac{1}{8N} \sum_{j=1}^{N} \sin^{-5} \frac{\pi(2j-1)}{2N}$$

the Hamiltonian near the equilibrium point is

$$H = \frac{1}{2} \{\omega_1^2 \rho^2 + \omega_2^2 z^2 + R^2 + Z^2\} - \omega_1^2 \rho^3 +$$
$$- \frac{3}{2} \omega_2^2 \rho z^2 + 3\omega_1^2 \rho^4 + 6\omega_2^2 \rho^2 z^2 - \frac{3}{4} (M + E_{2N}) Z^4 + \cdots .$$

From this we obtain

$$D_{2N} = \frac{9\omega_1^2}{16\omega_2^2} \left(-M - E_{2N} + \frac{3\omega_2^2}{4\omega_2^2 - \omega_1^2}\right).$$

It is easy to see that D_N is always nonzero and that D_{2N} is zero for at most one value of M for fixed N. When D_N or D_{2N} is nonzero Arnold's stability theorem applies to prove the existence of invariant two-dimensional tori near the equilibrium point. The solutions on these invariant tori are quasi-periodic. Also, one can show that the Liapunov periodic solutions are of general elliptic type and so Moser's invariant curve theorem shows that they are stable and Birkhoff's theorem shows that they are encircled by very long period periodic-orbits. These solutions differ from the solutions given by Liapunov's theorem in that both the r

and the z modes are excited together. That is, instead of being tangent to the r or z axis as are the Liapunov families, these periodic solutions are closed curves which encircle the origin in the r, z-plane. Returning to the full three dimensional x, y, z-space these solutions will be twisted closed curves which encircle the simple closed curve where $x^2 + y^2 = 1$, $z = 0$. In the nonalternating $(N + 1)$-body problem all the small bodies lie in a plane and so the ring looks like a circle which pulsates up and down while pulsating in and out. In the alternating $(2n + 1)$-body problem the small bodies follow two different curves in space which twist around the circle $x^2 + y^2 = 1$, $z = 0$. This solution might be called 'twisted' or 'braided' (see Figure 5).

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References

Davies, I., Truman, A. and Williams, D.: 1983, Classical periodic solutions of the equal-mass $2n$-body problem, $2n$-ion problem and the n-electron atom problem, *Phys. Lett.* **99A**, 15.

Davies, I., Truman, A. and Williams, D. (in prep.): Classical periodic orbits for equal-mass many body systems, preprint.

Deprit, A.: 1969, Canonical transformations depending on a small parameter, *Celest. Mech.* **1**, 12–30.

Guillemin, V. and Sternberg, A.: 1984, *Symplectic Techniques in Physics*, Cambridge Univ. Press, Cambridge.

Lissauer, J. and Peale, S.: 1986, The production of 'braids' in Saturn's F Ring, *Icarus* **67**, 358–374.

Marsden, J. and Weinstein, A.: 1974, Reduction on symplectic manifolds with symmetries, *Rep. Math. Phys.* **5**, 121–130.

Maxwell, J.C.: 1859, *On the Stability of the Motion of Saturn's Rings*, Maxmillan & Co., London.

Meyer, K.: 1973, Symmetries and integrals in mechanics, in M. Peixoto (ed.), *Dynamical Systems* Academic Press, New York, pp. 259–273.

Meyer, K.: 1981, Hamiltonian systems with a discrete symmetry, *J. Diff. Eq.* **41**, 228–238.

Meyer, K. and Hall, G.: 1991, *Introduction to Hamiltonian Dynamical Systems and the N-Body Problem*, Springer-Verlag, New York.

Moeckel, R.: 1992, Linear stability of relative equilibria with a dominate mass, preprint.

Smith, B. *et al.*: 1981, Encounter with Saturn: Voyager 1 image science results, *Science* **212**, 163–190.

Smith, B. *et al.*: 1982, A new look at the Saturn system: Voyager 2 images, *Science* **215**, 504–536.

Weinstein, A.: 1973, Normal modes for non-linear Hamiltonian systems, *Inv. Math.* **20**, 47–57.

Wintner, A.: 1944, *The Analytic Foundations of Celestial Mechanics*, Princeton University Press, Princeton, NJ.