THE DETERMINATION OF THE DERIVATIVES IN BROWN'S LUNAR THEORY* **

KENNETH R. MEYER and DIETER S. SCHMIDT

Dept. of Mathematical Sciences, University of Cincinnati, Cincinnati, Ohio, U.S.A.

Abstract. The fundamental matrix solution T for the variational equations of a Hamiltonian system is symplectic. We use this fact to complete T when it is only partially known. We discuss three cases. The last one gives an easy proof for the method invented by Brown in his lunar theory.

Consider a Hamiltonian function $H(x, y)$ of n degrees of freedom. Let $x(t, \alpha, \beta)$ and $y(t, \alpha, \beta)$ be the solution of the corresponding system of differential equations. The n dimensional vectors α and β contain parameters which are used to fix the initial conditions for x and y at some specific time, say $t = 0$. The parameters are called canonical if the Jacobian $\partial(x, y)/\partial(\alpha, \beta)$ is symplectic at $t = 0$. The solution matrix $T = \partial(x, y)/\partial(\alpha, \beta)$ to the variational system of differential equations is then symplectic for all times. This fact can be used to complete the solution matrix T with the help of one quadrature when T is only partially known.

Applications for this method can occur for example when α are action parameters and β are the conjugate angular parameters. In developing the formal solution to the original system of Hamiltonian differential equations it may have been convenient to replace some or all components of the parameter vector α by their numerical values but to keep the components of β as formal parameters.

A typical example is Brown's lunar theory. There the first component of α has been replaced by its numerical value from the beginning of the computation. The component under discussion is in Brown's notation n, the observed mean motion of the Moon, or m a value which is directly related to n. The decision to construct a semianalytical solution rather then an analytical one was made by Brown because the resulting series in m converge very poorly. Otherwise the number of terms in his already lengthy solution would probably have increased by a factor of 30.

Brown (1903) devised a method for finding the derivatives with respect to m. When Brown (1908) uses the method he writes on page 11: "The process given in the preceding paragraph is neither easy in theory nor easy for computation. But in the absence of any other method it had to be adopted".

We will show that Brown's method follows easily from the fact that T and T^t are symplectic matrices, in particular if modern notation is used. Furthermore Brown's formula is just one of several which we will derive. Although the theory on which the methods depend is easy we have to admit that the computations are still involved. Despite the fact that the final formulas look simple they hide many of the technical details which have to be overcome when they are used.

* Paper presented at the 1981 Oberwolfach Conference on Mathematical Methods in Celestial Mechanics.
** Dedicated to Victor Szebehely.

Celestial Mechanics **28** (1982) 201–207. 0008–8714/82/0282–0201 $01.05.

In the following let $\dot{z} = JSz$ be the system of variational differential equations for a given Hamiltonian function and a certain solution, J is the standard symplectic matrix, $S = D^2H$ is evaluated at the given solution $x(t, \alpha, \beta)$, $y(t, \alpha, \beta)$ and $z = \begin{pmatrix} x \\ y \end{pmatrix}$. We will discuss three cases:

(1) All of the derivatives with respect to β are known but none with respect to α.

(2) Same as (1) with the addition that $H = \frac{1}{2}|y|^2 - F(x)$.

(3) Only the derivatives with respect to α_1, the first component of α, is unknown and $H = \frac{1}{2}|y|^2 - F(x)$.

The above cases serve only as an illustration. Related formulas can be found when the assumptions are modified.

THEOREM 1. *Let the Hamiltonian function $H = H(x, y)$ of n degrees of freedom admit the solution vector $\begin{pmatrix} x(t, \alpha, \beta) \\ y(t, \alpha, \beta) \end{pmatrix}$. The vectors α and β consist of canonical parameters. Assume that the $2n$ by n matrix $C = \begin{pmatrix} x_\beta \\ y_\beta \end{pmatrix}$ is known but that the formal derivatives with respect to α can not be formed. Let $S = D^2H$ be evaluated at the solution. From now on let D denote the $2n$ by n matrix $D = JC(C^tC)^{-1}$. Compute the n by n matrix*

$$V = \int_0^t D^t(SD + J\dot{D})\,dt.$$

Then a symplectic fundamental matrix solution for the corresponding system of variational equations is given by

$$T = (D - CV, C)$$

Proof. By assumption C has rank n, satisfies $C^tJC = 0$ and C^tC is nonsingular. Since $D^tJD = 0$ and $C^tJD = -I$ it follows that $P = (D, C)$ is a symplectic matrix. We therefore have $P^{-1} = \begin{pmatrix} -C^tJ \\ D^tJ \end{pmatrix}$.

Change to new coordinates by $z = P\zeta$. The transformed variational equations are

$$\dot{\zeta} = P^{-1}(JSP - \dot{P})\zeta = \begin{pmatrix} C^tSD + C^tJ\dot{D} & 0 \\ -D^tSD - D^tJ\dot{D} & 0 \end{pmatrix}\zeta.$$

All submatrices in the last expression have size n by n. The one in the upper left corner is also zero as can be verified by differentiating $C^tJD = I$. With $\zeta = \begin{pmatrix} u \\ v \end{pmatrix}$

the above system reads

$$\dot{u} = 0$$

$$\dot{v} = -D^t(SD + J\dot{D})u.$$

It has the general solution

$$u = u_0, \qquad v = -Vu_0 + v_0 \tag{1}$$

where V is defined in the theorem and the symplectic matrix T of the theorem follows from it.

If the derivatives with respect to α are needed we have to select the initial conditions for (1) so that at one time the matrix T agrees with the derivatives. From the specification of the initial conditions for x and y in terms of α and β at time $t = 0$ we find directly $x_\alpha(0)$ and $y_\alpha(0)$. Let P_0 be the matrix P at time $t = 0$. The desired $2n$ by n submatrix of the Jacobian T is then

$$\begin{pmatrix} x_\alpha(t) \\ y_\alpha(t) \end{pmatrix} = (D - CV, C)P_0^{-1}\begin{pmatrix} x_\alpha(0) \\ y_\alpha(0) \end{pmatrix}.$$

THEOREM 2. *Let $H = \frac{1}{2}|\dot{x}|^2 - F(x)$ have the general solution $x(t, \alpha, \beta)$. It depends on 2 canonical parameter vectors α and β of dimension n each. If x_β is known we can compute*

$$x_\alpha = x_\beta\left(K - \int_0^t (x_\beta^t x_\beta)^{-1}\,\mathrm{d}t\right)$$

where K is a constant symmetric matrix.

Proof. The Jacobian $T = \partial(x, \dot{x})/\partial(\alpha, \beta)$ is a symplectic matrix.

This fact in terms of submatrices reads

$$x_\alpha^t \dot{x}_\alpha - \dot{x}_\alpha^t x_\alpha = 0 \tag{2a}$$

$$x_\alpha^t \dot{x}_\beta - \dot{x}_\alpha^t x_\beta = I \tag{2b}$$

$$x_\beta^t \dot{x}_\beta - \dot{x}_\beta^t x_\beta = 0. \tag{2c}$$

With the help of (2b) and (2c) we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(x_\beta^{-1}x_\alpha) &= -x_\beta^{-1}\dot{x}_\beta x_\beta^{-1}x_\alpha + x_\beta^{-1}\dot{x}_\alpha \\ &= -(x_\beta^t x_\beta)^{-1}(x_\beta^t \dot{x}_\beta x_\beta^{-1} x_\alpha - x_\beta^t \dot{x}_\alpha) \\ &= -(x_\beta^t x_\beta)^{-1}(\dot{x}_\beta^t x_\alpha - x_\beta^t \dot{x}_\alpha) \\ &= -(x_\beta^t x_\beta)^{-1}.\end{aligned}$$

When we integrate we obtain the formula of the theorem with K as a constant matrix. Since x_α has to satisfy (2a) K has to be symmetric. K is obtained from the given initial conditions and is $K = x_\beta^{-1}(0)x_\alpha(0)$. It can be determined differently when the solution to the Hamiltonian system $x(t)$ satisfies a symmetry condition like $x(t) = Rx(-t)$ where R is a diagonal matrix with $R^2 = I$. If $x_\alpha(t) = Rx_\alpha(-t)$ and $x_\beta(t) = -Rx_\beta(-t)$ holds then this symmetry condition implies that $K = 0$.

Extensions of the above theorem are possible to the Hamiltonian $H = \frac{1}{2}y^t Ay - F(x)$ and to a noncanonical set of parameters, provided that A is symmetric and invertible. For now let α and β be an arbitrary set of radial and angular parameters. If c represents the conjugate set of parameters to β then we assume that $c = c(\alpha)$ does not depend on β. Denote by $C = \partial c/\partial\alpha$ the n by n transformation matrix and by A^{-t} the transpose of the inverse of the matrix A. These two extensions modify the conditions (2) so that for $T = \partial(x, \dot{x})/\partial(\alpha, \beta)$ we have

$$T^t\begin{pmatrix} 0 & A^{-1} \\ -A^{-t} & 0 \end{pmatrix}T = \begin{pmatrix} 0 & C^t \\ -C & 0 \end{pmatrix}. \tag{3}$$

The formula of theorem 2 has then to be changed to

$$x_\alpha = X_\beta\Big(K - \int_0^t X_\beta^{-1} A\, X_\beta^{-t}\, \mathrm{d}t\Big)$$

with

$$X_\beta = x_\beta C^{-t}.$$

THEOREM 3. *Let* $H = \frac{1}{2}|\dot{x}|^2 - F(x)$ *have the solution* $x(t, \alpha, \beta)$ *which depends on canonical parameters. Assume that only the derivative with respect to* α_1 *is not available. Then* $\partial x/\partial\alpha_1$ *can be found by a quadrature.*

Proof. Again let $T = \partial(x, \dot{x})/\partial(\alpha, \beta)$ but now use the fact that T^t is also symplectic, that is $TJT^t = J$. In terms of components the last relation expresses the invariance of the Poisson brackets. Of these brackets the following are of interest:

$$\sum_{k=1}^n \frac{\partial x_i}{\partial\alpha_k}\frac{\partial\dot{x}_j}{\partial\beta_k} - \frac{\partial x_i}{\partial\beta_k}\frac{\partial\dot{x}_j}{\partial\alpha_k} = \delta_{ij} \qquad i, j = 1, \ldots, n$$

For $i = j$ the Poisson bracket contains only $\partial x_i/\partial\alpha_1$ and $\partial\dot{x}_i/\partial\alpha_1$ as unknown quantities. After dividing by $(\partial x_i/\partial\beta_1)^2$ we can integrate and obtain

$$\frac{\partial x_i}{\partial\alpha_i} = \frac{\partial x_i}{\partial\beta_i}\left\{K + \int_0^t\left[\sum_{k=2}^n\left(\frac{\partial x_i}{\partial\alpha_k}\frac{\partial\dot{x}_i}{\partial\beta_k} - \frac{\partial x_i}{\partial\beta_k}\frac{\partial\dot{x}_i}{\partial\alpha_k}\right) - 1\right]\left(\frac{\partial x_i}{\partial\beta_1}\right)^{-2}\mathrm{d}t\right\}.$$

The constant K can be determined from symmetry considerations or by using a Poisson bracket with $i \neq j$.

This proof replaces in essence the first part of Brown's (1903) paper. The rest of his paper discusses the application to the lunar problem. We will outline this now briefly. The first step requires the same generalization for theorem 3 as was given for theorem 2. As the notation for the general case becomes cumbersome we will restrict outselves to $n = 3$ in the way it is needed for the lunar problem.

Let the position vector be $x = (u, v, z)^t$ with u and v as complex coordinates for the x_1, x_2 plane and with z perpendicular to them. The corresponding momenta vector is $y = (U, V, Z)^t$ so that

$$H = 2UV + \tfrac{1}{2}Z^2 - F(u, v, z). \tag{4}$$

If H were transformed locally to canonical action angle variables (c_1, c_2, c_3) and (w_1, w_2, w_3) respectively it would be a function of the c_k's only. The corresponding differential equations would be

$$\dot{c}_k = 0$$

$$\dot{w}_k = -\frac{\partial H}{\partial c_k} = b_k(c_1, c_2, c_3) \qquad k = 1, 2, 3.$$

The general solution to (4) can thus be given by a complex Fourier series of the form

$$u = \sum A_j \exp i(j_1 w_1 + j_2 w_2 + j_3 w_3). \tag{5}$$

The summation is over all integer triples $j = (j_1, j_2, j_3)$. We have $A_j = A_j(c_1, c_2, c_3)$ and $w_k = b_k(c_1, c_2, c_3)t + \beta_k$. Furthermore v is the complex conjugate to u. A similar expression holds for z.

In the lunar problem b_1 is the mean motion of the Moon and it is replaced by its numerical value n from the outset. Therefore $\alpha_1 = n$ is not a canonical variable. The relationship to the canonical set is

$$n = b_1(c_1, c_2, c_3)$$

$$\alpha_2 = c_2$$

$$\alpha_3 = c_3.$$

For the inverse transformation we compute

$$C = \frac{\partial c}{\partial \alpha} = \begin{pmatrix} \dfrac{\partial c_1}{\partial n} & \dfrac{\partial c_2}{\partial \alpha_2} & \dfrac{\partial c_3}{\partial \alpha_3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Instead of $TJT^t = J$ we have to use a more general condition which follows directly from (3):

$$T\begin{pmatrix} 0 & C^{-1} \\ -C^{-t} & 0 \end{pmatrix}T^t = \begin{pmatrix} 0 & A^t \\ -A & 0 \end{pmatrix}. \tag{6}$$

Among the different relations given by (6) we select the one which involves only u_{α_1} and $\dot{u}_{\alpha_1}$ as unknown quantities. In vector notation this is given by

$$u_\alpha C^{-1}\dot{u}^t_\beta - \dot{u}_\alpha C^{-1}u^t_\beta = 0. \tag{7}$$

All frequencies in (5) depend on α. For the perturbation theory later on the change of the coefficients A_j with respect to α is of interest. Therefore Brown has introduced the following convention: u_α will denote the partial derivative with respect to α whereas $\partial u/\partial\alpha$ indicates that only the coefficients A_j in (5) are to be differentiated. With this notation we find

$$u_\alpha = \frac{\partial u}{\partial\alpha} + tu_\beta\frac{\partial b}{\partial\alpha} \tag{8a}$$

and

$$\dot{u}_\alpha = \frac{\partial}{\partial\alpha}\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right) + t\dot{u}_\beta\frac{\partial b}{\partial\alpha}. \tag{8b}$$

The vector b is made up of the different frequencies b_k. Note that $\partial/\partial\alpha$ and $\mathrm{d}/\mathrm{d}t$ do not commute, as we have for example

$$\frac{\partial}{\partial\alpha_1}\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial u}{\partial\alpha_1} + u_\beta\frac{\partial b}{\partial\alpha_1} \tag{9}$$

When (8a, b) is used in (7) the terms in t vanish since

$$\frac{\partial b}{\partial\alpha}C^{-1} = \frac{\partial b}{\partial c}$$

is a symmetric matrix. For the same reason $C^t\ \partial b/\partial\alpha$ is symmetric. This gives the following two identities $\partial b_2/\partial n = -\partial c_1/\partial\alpha_2$ and $\partial b_3/\partial n = -\partial c_1/\partial\alpha_3$. They are used to rewrite the last term in (9) which we then rename to be

$$U = u_\beta\frac{\partial b}{\partial\alpha_1} = u_{\beta_1} - \frac{\partial c_1}{\partial\alpha_2}u_{\beta_2} - \frac{\partial c_1}{\partial\alpha_3}u_{\beta_3}. \tag{10}$$

Finally if the terms in (7) without t are written down with the help of (8), (9), and (10) we find

$$\frac{\partial u}{\partial n}\dot{U} - \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial u}{\partial n} + U\right)U + \frac{\partial c_1}{\partial n}Q = 0$$

where

$$Q = \frac{\partial u}{\partial \alpha_2}\frac{\partial \dot{u}}{\partial \beta_2} - \frac{\partial \dot{u}}{\partial \alpha_2}\frac{\partial u}{\partial \beta_2} + \frac{\partial u}{\partial \alpha_3}\frac{\partial \dot{u}}{\partial \beta_3} - \frac{\partial \dot{u}}{\partial \alpha_3}\frac{\partial u}{\partial \beta_3}.$$

After dividing by U^2 we can integrate and we obtain the formula given by Brown

$$\frac{\partial u}{\partial n} = U\left(K + \int_0^t \left(\frac{\partial c_1}{\partial n}\frac{Q}{U^2} - 1\right)\mathrm{d}t\right) \tag{11}$$

Since no secular terms can appear $\partial c_1/\partial n$ must be the reciprocal of the constant term in the expression of QU^{-2}.

Symmetry conditions only tell that K has to be a purely imaginary constant. Its value has to be determined by another relation of (6), for example by

$$u_\alpha C^{-1} v^t_\beta - v_\alpha C^{-1} u^t_\beta = 0.$$

If we call Φ the integral in (11) then the above condition reads

$$\mathrm{Im}\left\{|U|^2(K + \Phi) + \frac{\partial c_1}{\partial n}\left(\frac{\partial u}{\partial \alpha_2}\frac{\partial v}{\partial \beta_2} + \frac{\partial u}{\partial \alpha_3}\frac{\partial v}{\partial \beta_3}\right)\right\} = 0.$$

Although this is a relation among series it suffices to determine K from the constant terms. The rest of the series and additional relations in (6) can be used to check the accuracy of the computations.

The derivative $\partial z/\partial n$ is computed in an analogous way, but we will not reproduce the method of Brown for it here.

The above method for finding the derivative with respect to n is implemented by the second author for the solution to the main problem which was given in Schmidt (1980).

Acknowledgement

This research was supported by the National Science Foundation grant MCS 80–01851.

References

Brown, E. W.: 1903, *Trans. Am. Math. Soc.* **4**, 234.
Brown, E. W.: 1908, *Mem. Roy. Astron. Soc.* **59**, 1.
Schmidt, D. S.: 1980, *The Moon and the Planets* **23**, 135.