PERIODIC ORBITS NEAR INFINITY IN THE RESTRICTED N-BODY PROBLEM

KENNETH R. MEYER*
Department of Mathematics, University of Cincinnati, Ohio, U.S.A.

(Received June, 1979; Accepted October, 1979)

Abstract. This paper shows that there exist two families of periodic solutions of the restricted N-body problem which are close to large circular orbits of the Kepler problem. These solutions are shown to be of general elliptic type and hence are stable. If the restricted problem admits a symmetry, then there are symmetric periodic solutions which are close to large elliptic orbits of the Kepler problem.

1. Introduction

This paper investigates the existence and stability of periodic solutions of the restricted N-body problem where the infinitesimal body is at a great distance from the center of mass of the primaries. Moulton (1912) established the existence of periodic solutions of the restricted three-body problem which are symmetric with respect to the line of masses of the primaries and are nearly circles of large radii. Moulton's proof is simple and elegant, but he only proves the existence of these periodic solutions. Below we present an elementary proof of the existences which gives as a by-product an estimate on the characteristic multipliers. A second proof of the existence shows that these solutions are of general elliptic type. Since these solutions are of general elliptic type, a theorem of Birkhoff (1927) can be applied to show that close to these nearly circular orbits are periodic solutions of very long period and the theorems of Arnold (1961) and Moser (1962) can be applied to show that these nearly circular orbits are stable. Since neither of the proofs presented here depend on any special symmetry property of the Hamiltonian of the system, they apply to the restricted N-body problem which is not symmetric in general. However, if the Hamiltonian of the restricted N-body problem does admit a discrete symmetry, we show that the equations of motion have symmetric periodic orbits which are close to large elliptic periodic orbits of a related Kepler problem. These nearly elliptic orbits are established by using the methods and implicit function theorem of Arenstorf (1966, 1968, 1978) and Arenstorf and Bozeman (1977).

In Section 2 it is shown that the Hamiltonian of the restricted N-body problem is a limiting case of the full N-body problem as one of the masses tends to zero. This derivation clarifies the previously tenuous connection between the restricted and the full N-body problem and obviates which periodic orbits can be continued into the full problem. All subsequent sections discuss the restricted problem only.

* This research partially supported by NSF grant NCS 75-05862.

Celestial Mechanics **23** (1981) 69–81. 0008–8714/81/0231–0069 $01.95.

In Section 3 we show that the Kepler problem in rotating coordinates is a limiting case of the restricted N-body problem as the distance from the center of mass of the primaries to the infinitesimal body tends to infinity.

In Section 4 we use a simple lemma from non-linear oscillation theory to establish the existence of two families of periodic orbits which are close to circles of very large radii. In an inertial coordinate system one family is direct and one is retrograde. This lemma and a simple calculation shows that both families consist of elliptic periodic solutions. That is, we show that the non-trivial characteristic multipliers of these periodic solutions are of unit modulus and not equal to ± 1.

In Section 5 we consider those restricted N-body problems which have a line of symmetry (for example, the restricted three-body problem). Delaunay elements are used with the implicit function theorem of Arensdorf to establish the existence of symmetric periodic solutions of the restricted N-body problem which are close to very large elliptic orbits of the Kepler problem.

In Section 6, Poincaré variables are used to give an alternate proof of the existence of the nearly circular orbits near infinity. In these coordinates it is easy to prove that these nearly circular orbits are of general elliptic type in that they satisfy the 'twist condition' of Birkhoff (1927), Arnold (1961) and Moser (1962). Thus, close to these nearly circular orbits are other periodic orbits of very long period and quasi-periodic orbits filling invariant tori. This also implies that these nearly circular orbits are stable.

2. The Restricted Problem – A Limiting Case

In the classical, and now universal derivation, of the restricted N-body problem one is asked to consider the motion of a body of infinitesimal mass which is subject to the gravitational attraction of $N-1$ other bodies which move on a central configuration solution. One is to assume that the infinitesimal body does not exert any influence on the motion of the $N-1$ massive bodies and so the massive bodies always move on a central configuration solution. From this description it is clear how to write the Hamiltonian of the problem and it is also clear that the Hamiltonian is somehow related to the Hamiltonian of the full N-body problem. However, the precise connection between the restricted and the full N-body problem is never quite clear in the classical derivation.

Here we shall obviate the connection by scaling the Hamiltonian of the N-body problem by a symplectic change of variables which depends on a small parameter ε which is one of the masses. In the new scaled variables the Hamiltonian decouples to lowest order in the small parameter ε into two terms. The first term is the Hamiltonian of the restricted N-body problem and the second term is a quadratic form in the remaining variables. The equations of motion obtained from these quadratic terms are the linearized equations of motion of the $(N-1)$-body problem about the relative equilibrium.

In this paper only the planar, circular restricted N-body will be considered. Let $q_1, \cdots, q_N \in \mathbb{R}^2$ be the position vectors of N-bodies in the plane with masses $m_1, \ldots, m_N$ and momenta $p_1, \ldots, p_N$ respectively. The Hamiltonian of the N-body problem is

$$H_N = \sum_{i=1}^{N} \frac{\|p_i\|^2}{2m_i} - \sum_{1 \le i < j \le N} \frac{m_i m_j}{\|q_i - q_j\|} \tag{2.1}$$

and the equations of motion are

$$\dot{q}_i = p_i/m_i, \qquad \dot{p}_i = \sum_{j=1}^{N}{}' \frac{m_i m_j (q_j - q_i)}{\|q_j - q_i\|^3}, \quad i = 1, \ldots, N. \tag{2.2}$$

Here and below the prime on the summation sign indicates that the term where $i = j$ is excluded. These equations have periodic solutions where the bodies move on concentric circles with uniform velocity. If the origin is taken as the center of mass, then these periodic solutions must be of the form

$$q_i^* = e^{-\omega J t} a_i, \qquad p_i^* = -m_i \omega J\, e^{-\omega J t} a_i, \quad i = 1, \ldots, N, \tag{2.3}$$

where $a_1, \ldots, a_N$ are constant vectors, ω is a positive number (the frequency) and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

so

$$e^{\omega J t} = \begin{pmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{pmatrix}.$$

In order for q_i^* and p_i^* to be solutions of (2.2) the a_i and ω must satisfy the non-linear algebraic equations

$$\omega^2 a_i + \sum_{j=1}^{N}{}' \frac{m_j (a_j - a_i)}{\|a_j - a_i\|^3} = 0, \quad i = 1, \ldots, N. \tag{2.4}$$

The geometric configuration of the N-bodies given by $a_1, \ldots, a_N$ is called a central configuration and the solution (2.3) is called a central configuration solution. For the three-body problem the only central configurations are the equilateral triangle configurations of Lagrange and the collinear configurations of Euler (see Siegel, 1956, for a thorough discussion). Moulton (1912) and Smale (1970) have shown that up to similarity transformations there are $N!/2$ collinear configurations of the N-body problem. Many other central configurations are known, but no complete classification is known for $N > 3$. We shall show that for each central configuration there is a corresponding restricted problem.

If the equations are changed to a rotating coordinate system which rotates with constant angular frequency ω, then these central configuration solutions will become equilibrium solutions and therefore the solutions (2.3) are also called solutions of a relative equilibrium. Introducing rotating coordinates by $q_i = e^{-\omega Jt}x_i$, $p_i = e^{-\omega Jt}y_i$ the Hamiltonian becomes

$$H_N = \sum_{i=1}^{N} \left\{ \frac{\|y_i\|^2}{2m_i} - \omega x_i^T J y_i \right\} - \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{\|x_i - x_j\|} \tag{2.5}$$

and the equations of motion are

$$\dot{x}_i = y_i / m_i + \omega J x_i , \qquad \dot{y}_i = \omega J y_i + \sum_{j=1}^{N}{}' \frac{m_i m_j (x_j - x_i)}{\|x_j - x_i\|^3} . \tag{2.6}$$

Henceforth we shall always assume $\omega = 1$, since this can always be accomplished by a change in the time scale. Let

$$J_k = \begin{pmatrix} 0_k & I_k \\ -I_k & 0_k \end{pmatrix}$$

where 0_k is the $k \times k$ zero matrix and I_k is the $k \times k$ identity matrix (so $J_1 = J$ of our previous notation). Let $Z = (x_1^T, \ldots, x_N^T, y_1^T, \ldots, y_N^T)^T$ and $Z^* = (a_1^T, \ldots, a_N^T, -m_1 a_1^T J, \ldots, - m_N a_N^T J)^T$ so that equations (2.6) become

$$\dot{Z} = J_{2N} \nabla H_N(Z) \tag{2.7}$$

and Z^* is the relative equilibrium solution; thus $\nabla H_N(Z^*) = 0$. Since Z^* is a critical point for H, one has by Taylor's theorem

$$H_N(Z) = H_N(Z^*) + \tfrac{1}{2}(Z - Z^*)^T S (Z - Z^*) + 0(\|Z - Z^*\|^3) , \tag{2.8}$$

where $S = (\partial^2 H / \partial Z^2)(Z^*)$ is the Hessian of H at Z^*. The linearized equations of motion about the relative equilibrium solution Z^* are

$$\dot{Z} = J_{2N} S Z . \tag{2.9}$$

Equations (2.9) are the equations of the first approximation of the full equations (2.7) for solutions near the relative equilibrium.

Now let us assume that one mass is small by setting $m_N = \varepsilon^2$ and considering ε as a small parameter. Then the Hamiltonian becomes

$$H_N = \frac{\|y_N\|^2}{2\varepsilon^2} - x_N^T J y_N - \sum_{j=1}^{N-1} \frac{\varepsilon^2 m_j}{\|x_j - x_N\|} + H_{N-1} . \tag{2.10}$$

Let $x_1 = a_1, \ldots, x_{N-1} = a_{N-1}$, $y_1 = m_1 J a_1, \ldots, y_{N-1} = m_{N-1} J a_{N-1}$ be a relative equilibrium for the $(N-1)$-body problem whose Hamiltonian is H_{N-1} in (2.10). Introduce coordinates for this $(N-1)$-body problem as above so now let $Z = (x_1^T, \ldots, x_{N-1}^T, y_1^T, \ldots, y_{N-1}^T)^T$ and let $Z^* = (a_1^T, \ldots, a_{N-1}^T, -m_1 a_1^T J, \ldots, - m_{N-1} a_{N-1}^T J)^T$. Thus H_{N-1} is of the same form as H_N in (2.8). Make the change of variables

$$x_N = u , \qquad y_N = \varepsilon^2 v , \qquad Z = Z^* - \varepsilon U . \tag{2.11}$$

in (2.10). This change of variables is symplectic with multiplier ε^{-2} and so the Hamiltonian becomes

$$H_N = \left(\frac{\|v\|^2}{2} - u^T Jv - \sum_{j=1}^{N-1} \frac{m_j}{\|a_j - u\|}\right) + \tfrac{1}{2} U^T SU + O(\varepsilon) \tag{2.12}$$

plus a constant term $\varepsilon^{-2} H_{N-1}(Z^*)$ which we will ignore. Thus, to zeroth order in the small parameter ε the Hamiltonian is the sum of

$$H = \frac{\|v\|^2}{2} - u^T Jv - \sum_{j=1}^{N-1} \frac{m_j}{\|a_i - u\|} \tag{2.13}$$

and

$$K = \tfrac{1}{2} U^T SU. \tag{2.14}$$

The Hamiltonian H of (2.13) is the Hamiltonian of an infinitesimal body whose position is u and momentum is v in a rotating coordinate system and moves under the attraction of $N-1$ bodies of mass m_i at position a_i (the primaries). The Hamiltonian K of (2.14) is the Hamiltonian of the linearized problem of $N-1$ bodies about the relative equilibrium solution Z^*.

This decomposition shows that it is necessary to make certain assumptions about S if one hopes to continue a periodic solution from the restricted problem into the full N-body problem. The author in Meyer (1980) has investigated this question and has given sufficient conditions for the continuation of a periodic solution.

A corollary of one of these results is the theorem of Hadjidemetriou (1975) which is applicable to the work given below. Namely:

THEOREM. *Let* $u = \phi(t)$, $v = \psi(t)$ *be a T-periodic solution of the restricted three-body problem with characteristic multipliers* $1, 1, \beta, \beta^{-1}$. *If* $\beta \neq 1$ *and* $T \not\equiv 0 \bmod 2\pi$ *then this periodic solution can be continued into the three-body problem. That is, there is a* $\tilde{T}(\varepsilon)$*-periodic solution* $u = \tilde{\phi}(t, \varepsilon)$, $v = \tilde{\psi}(t, \varepsilon)$, $U = U(t, \varepsilon)$ *of the three-body problem (with Hamiltonian* (2.12)) *for small* ε *such that* $\tilde{T}(\varepsilon) \to T$, $\tilde{\phi}(t, \varepsilon) \to \phi(t)$, $\tilde{\psi}(t, \varepsilon) \to \psi(t)$, $\tilde{U}(t, \varepsilon) \to 0$ *as* $\varepsilon \to 0$.

3. Equations for Orbits at Infinity

Henceforth we shall consider only the restricted N-body problem as given in the previous section and we shall assume that the sum of the masses of the primaries is 1. Thus, the Hamiltonian under consideration is

$$H = \frac{\|v\|^2}{2} - u^T Jv - \sum_{j=1}^{N-1} \frac{m_j}{\|a_j - u\|}, \tag{3.1}$$

where $m_1 + \cdots + m_{N-1} = 1$ and the equations of motion are

$$\dot{u} = v + Ju, \qquad \dot{v} = Jv + \sum_{j=1}^{N-1} \frac{m_j(a_j - u)}{\|a_j - u\|^3}. \tag{3.2}$$

In order to study this problem for large values of u we introduce a small positive scale parameter μ and make the change of scale $u = \mu^{-2}\xi$, $v \doteq \mu\eta$. This is a symplectic change of variables with multiplier μ and so the Hamiltonian becomes

$$H = \mu\left(\frac{\mu^2\|\eta\|^2}{2} - \mu^{-1}\xi^T J\eta - \sum_{j=1}^{N-1} \frac{\mu^2 m_j}{\|\mu^2 a_j - \xi\|}\right)$$
$$= -\xi^T J\eta + \mu^3(\|\eta\|^2/2 - 1/\|\xi\|) + O(\mu^5)\,. \tag{3.3}$$

The equations of motion are

$$\begin{aligned} \dot{\xi} &= J\xi + \mu^3\eta + O(\mu^5) \\ \dot{\eta} &= J\eta - \mu^3\xi/\|\xi\|^3 + O(\mu^5)\,. \end{aligned} \tag{3.4}$$

To lowest order in μ these equations are linear and have the general solution $\xi = \exp(Jt)\xi_0$, $\eta = \exp(Jt)\eta_0$ which are 2π-periodic in t. Thus for small μ the solutions of (3.4) are approximately circular. Therefore for large initial values u and small initial values v the solutions of (3.2) are approximately circular. Since rotating coordinates are being used this means that near infinity the infinitesimal body mainly feels the effect of the Coriolis and centrifugal forces and in a fixed coordinate system would be approximately at rest. The coefficient of the μ^3 term is the Hamiltonian of the Kepler problem where the central body has mass 1 and is located at the origin. This can be interpreted as meaning that the next most important force felt by the infinitesimal body when it is near infinity is the attraction of a fixed body at the center of mass of the primaries whose mass is equal to the sum of the masses of the primaries.

The equations of the restricted N-body problem have singularities at the primaries and sometimes this is hidden in the $O(\mu^5)$ term. However, from (3.3) we see that these singularities are at $\mu^2 a_j$, $j = 1, \ldots, N-1$ and so tend to the origin as $\mu \to 0$. Thus, there is no problem in avoiding the singularities in the analysis that follows.

4. Circular Orbits at Infinity

The Equations (3.4) are of the form studied in non-linear oscillation theory (see for example Chapter 14 of Coddington and Levinson (1955) or Chapter 5 of Hale (1969)). They are non-linear, contain a small parameter μ, and reduce to a linear system when $\mu = 0$. However, this system is somewhat degenerate since it is autonomous, it admits an integral and all its solutions are 2π periodic when $\mu = 0$. These degeneracies can be easily overcome by a careful application of the implicit function theorem. Indeed, the technical lemma of Meyer and Schmidt (1971) is adequate for the present problem and we shall summarize a simplified version of this lemma here.

Consider the system

$$\dot{\zeta} = A\zeta + \mu^3 f(\zeta, \mu)\,, \tag{4.1}$$

where $\zeta \in \mathbb{R}^n$, $\mu \in \mathbb{R}$, A is an $n \times n$ constant matrix such that $\exp A2\pi = I$ and F is analytic in an open set of the form $Q \times \mathbb{R}$ where Q is open in $\mathbb{R}^n$. Assume that (4.1) admits an integral of the form $I(\zeta, \mu) = \zeta^T D\zeta + O(\mu^3)$ where I is analytic on $Q \times \mathbb{R}$ and D is an $n \times n$ symmetric matrix. Let β be a real parameter and define

$$B(\beta, \zeta) = \beta A\zeta + \int_0^{2\pi} e^{-As} f(e^{As}\zeta, 0)\, ds\,. \tag{4.2}$$

Remark. The equations $B(\beta, \zeta) = 0$ are often called the bifurcation or determining equations in non-linear oscillation theory.

LEMMA. *If there exists a $\xi_0 \in Q$ and a $\beta_0 \in \mathbb{R}$ such that*

$$B(\beta_0, \zeta_0) = 0\,,$$

$$D\zeta_0 \neq 0\,,$$

$$\operatorname{rank}\left(\frac{\partial B}{\partial \beta}, \frac{\partial B}{\partial \zeta}\right)(\beta_0, \zeta_0) = n - 1\,,$$

then there exists an analytic one-parameter family of periodic solutions of (4.1), *denoted by $\phi(t, \mu)$, such that*

$\phi(t, \mu)$ is defined for all μ, $|\mu| \leq \mu_0$ and periodic in t of period $2\pi + \mu^3\beta_0 + O(\mu^4)$,

$\phi(t, 0) = (\exp At)\zeta_0$,

the characteristic multipliers of $\phi(t, \mu)$ are $1 + \mu^3\delta_i + O(\mu^4)$ where δ_i are the eigenvalues of $(\partial B/\partial \zeta)(\beta_0, \zeta_0)$.

A similar lemma is found in Hale (1969).

To apply this lemma to Equations (3.4) we set $\zeta = (\xi^T, \eta^T)^T$, $\zeta_0 = (\xi_0^T, \eta_0^T)^T$, $A = \operatorname{diag}(-J, -J)$, $f(\zeta, \mu) = (\eta^T, -\xi^T/\|\xi\|^3)^T + O(\mu^2)$. Since $\|\exp -Jt\xi_0\| = \|\xi_0\|$ the integral in (4.2) reduces to integrating a constant and so trivially the bifurcation equations are

$$\begin{aligned} &\beta J\xi_0 + 2\pi\eta_0 = 0\,, \\ &\beta J\eta_0 - 2\pi\xi_0/\|\xi_0\|^3 = 0\,. \end{aligned} \tag{4.3}$$

Solving the first equation for η_0 and substituting into the second yields

$$\left(\beta^2 - \frac{(2\pi)^2}{\|\xi_0\|^3}\right)\xi_0 = 0\,. \tag{4.4}$$

For any ξ_0 this equation can be solved by taking $\beta = \pm 2\pi/\|\xi_0\|^3$. Take $\xi_0 = (1, 0)^T$ and so $\beta = \pm 2\pi$ and $\eta_0 = (0, \mp 1)^T$. The Jacobian $\partial B/\partial \zeta$ at this solution is

$$2\pi \begin{pmatrix} 0 & \mp 1 & 2 & 0 \\ \pm 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & \mp 1 \\ 0 & 1 & \pm 1 & 0 \end{pmatrix}$$

which is clearly of rank 3 and has characteristic equation $\lambda^2(\lambda^2+1)$. The integral for Equations (3.4) is the Hamiltonian (3.3) and

$$D=\frac{1}{2}\begin{pmatrix}0 & J\\ J & 0\end{pmatrix}$$

so clearly $D\zeta_0 \neq 0$. Thus the conditions of the lemma apply to both solutions $\beta_0 = 2\pi$, $\xi_0=(1,0)^T$, $\eta_0=(0,-1)^T$ and $\beta_0=-2\pi$, $\xi_0=(1,0)^T$, $\eta_0=(0,1)^T$. Thus Equations (3.4) have two families of periodic solutions for μ small, one of period $2\pi+\mu^3 2\pi + O(\mu^4)$ and the other of period $2\pi-\mu^3 2\pi+O(\mu^4)$. Both families have characteristic multipliers $1, 1, 1+\mu^3 i+O(\mu^4), 1-\mu^3 i+O(\mu^4)$ and hence are elliptic. As μ tends to zero these solutions tend to circular orbits. In the non-rotating coordinate system one family is direct and one family is retrograde.

Thus the restricted N-body problem has two families of elliptic periodic orbits which are close to circular orbits of very large radius.

5. Nearly Elliptic Orbits

Some central configurations are symmetric in that they are invariant under a reflection in a line. The collinear configurations of Euler and Moulton are invariant under a reflection in the line through the bodies, and the equilateral triangle configuration of Lagrange in the three-body problem with equal masses is invariant under a reflection in a median of the traingle. In this section we shall exploit the symmetry to establish additional periodic solutions which are nearly elliptic.

Assume that the central configuration giving rise to the restricted problem is symmetric and that the coordinate systems of Section 3 are so chosen that the line of symmetry is the abscissa in position space. This is the usual choice of coordinates for the restricted three-body problem. Then the Hamiltonian (3.3) is invariant under the anti-symplectic involution

$$\begin{aligned}&\xi_1\to\xi_1, \qquad \eta_1\to-\eta_1,\\ &\xi_2\to-\xi_2, \qquad \eta_2\to\eta_2,\end{aligned} \tag{5.1}$$

where $\xi=(\xi_1,\xi_2)^T$ and $\eta=(\eta_1,\eta_2)^T$. In this case an easy and classical argument shows that if a solution crosses the line of symmetry orthogonally at times 0 and $T>0$, then this solution is $2T$-periodic and the orbit is symmetric (see Birkhoff, 1927). That is, if $\xi=[\phi_1(t),\phi_2(t)]^T$, $\eta=[\psi_1(t),\psi_2(t)]^T$ is a solution of (3.4) such that

$$\begin{aligned}&\phi_2(0)=0, \qquad \psi_1(0)=0,\\ &\phi_2(T)=0, \qquad \psi_1(T)=0,\end{aligned} \tag{5.2}$$

where $T>0$ then this solution is $2T$-periodic. Many authors including Birkhoff and Moulton have used this fact to prove the existence of periodic solutions when the equally simple arguments (used in Section 2) give a more general existence theorem with additional information. In many cases, especially those considered by Arenstorf (1966, 1968, 1978), the method of the previous section fails and it is necessary to use the symmetry property. Here we consider a case where the symmetry is essential to the proof of the existence of the periodic solution and therefore we use the methods developed by Arenstorf.

If the $O(\mu^5)$ terms in (3.3) are not present, then (3.3) is the Hamiltonian of the Kepler problem in rotating coordinates. We shall show that some of the symmetric elliptic periodic solutions of this Kepler problem can be continued into the restricted problem. In order to ease the calculations Delaunay's elements l, g, L, G are introduced. First make the symplectic change of coordinates from (ξ, η) to (r, θ, R, Θ) where (r, θ) are polar coordinates in the ξ-plane and (R, Θ) are their conjugate momentum. The transformation is given by

$$\begin{aligned} \xi_1 &= r \cos\theta, \qquad & \eta_1 &= R\cos\theta - (\Theta/r)\sin\theta, \\ \xi_2 &= r \sin\theta, \qquad & \eta_2 &= R\sin\theta - (\Theta/r)\cos\theta, \end{aligned} \tag{5.3}$$

and so the Hamiltonian (3.3) becomes

$$H = -\Theta + \mu^3 \tfrac{1}{2}(R^2 + \Theta^2/r^2) - 1/r + O(\mu^5). \tag{5.4}$$

Next introduce Delaunay's elements by the sympletic change of variables which is generated by the function

$$W(r, \theta, L, G) = G + \int_{L(L-(L^2-G^2)^{1/2}]}^{r} \left(-\frac{G^2}{\rho^2} + \frac{2}{\rho} - \frac{1}{L^2}\right)^{1/2} \mathrm{d}p. \tag{5.5}$$

G is angular momentum, l is the mean anomaly, and g is the argument of the perihelion; both l and g are angular variables and so are defined modulo 2π. In these coordinates

$$H = -G - \mu^3/(2L^2) + O(\mu^5). \tag{5.6}$$

Delaunay's elements (l, g, L, G) are a valid set of coordinates in the domain in phase space which is the union of the elliptic solutions of Kepler's problem. In these coordinates an orthogonal crossing of the line of symmetry occurs when both l and g are integer multiples of π. See Szebehely (1967) for a complete discussion of Delaunay's elements and the symmetry condition.

The equations of motion are

$$\begin{aligned} \dot{l} &= \mu^3/L^3, \qquad & \dot{L} &= 0, \\ \dot{g} &= -1, \qquad & \dot{G} &= 0, \end{aligned} \tag{5.7}$$

plus terms $O(\mu^5)$. These equations are autonomous and so we may take the fast angle g as the independent variable so that the equations become

$$\frac{\mathrm{d}l}{\mathrm{d}g} = \frac{-\mu^3}{L^3}, \qquad \frac{\mathrm{d}L}{\mathrm{d}g} = 0, \qquad \frac{\mathrm{d}G}{\mathrm{d}g} = 0, \tag{5.8}$$

plus terms $O(\mu^5)$. For the moment ignore the $O(\mu^5)$ terms and seek a symmetric periodic solution of the approximate equations. Let a and b be relatively prime integers and set $\mu^3 = a/b$. Start with initial conditions $L = 1, G = 1, L = \pi$ and integrate on g from π to $(1+b)\pi$ to obtain the approximate solution

$$l = \pi - \mu^3 b\pi = (1-a)\pi , \qquad L = 1, \qquad G = 1 . \tag{5.9}$$

Thus, this approximate solution satisfies the symmetry condition and so to this level of approximation is a symmetric period solution.

By fixing a and taking b large the scale parameter μ is small and so one might expect that these approximate solutions can be continued into the restricted problem. However, the problem is complicated by the fact that taking b large corresponds to integrating the equations over a large variation of g. As Arenstorf has observed, the usual implicit function theorem cannot be applied since one cannot set $\mu = 0$ and find the approximate solution. Thus we must follow Arenstorf and make careful estimates.

First, we shall fix the integer a and the initial condition for G once and for all. Let the subscript f denote the full solution of (5.8), the subscript a the approximate solution of (5.8), and the subscript e the error term. Integrate the full Equation (5.8) with L_0, $l = \pi$ when $g = \pi$ to $g = (1+b)\pi$ to obtain

$$l_f(b, \mu, L_0) = l_a(b, \mu, L_0) + l_e(b, \mu, L_0) , \tag{5.10}$$

where

$$l_a(b, \mu, L_0) = \pi - \mu^3 b\pi / L_0^3 . \tag{5.11}$$

The error term l_e is due to the $O(\mu^5)$ terms which must be added to (5.8). The error term is $O(\mu^5)$ and the Lipshitz constant for the equations is $O(\mu^3)$ and so by the standard Grownwall estimate (see Hartman, 1964)

$$|l_e| \leq c_1 \mu^5 (e^{c_2 \mu^3 b} - 1) , \tag{5.12}$$

where c_1 and c_2 are constants. In this estimate the solutions must lie in a compact neighborhood of the approximate solution. A similar estimate holds on the first partials of l_e.

The approximate equation has a solution of $l_a(b, \mu, L_0) = (1-a)\pi$ by taking $\mu^3 = a/b$ and $L_0 = 1$. Also at this solution $\partial l_a / \partial L_0 = 3a\pi$ which is a fixed non-zero number. From the estimate (5.12) the error term can be made arbitrarily small by taking b large and fixing $\mu^3 = a/b$ since in this case the estimate (5.12) reads $|l_e| \leq c_1 (a/b)^{5/3} (e^{c_2 a} - 1)$. Similarly the derivatives of l_e can be made small by taking b large. These estimates assure that we remain in a compact neighborhood of the approximate solution. Thus, the implicit function theorem of Arenstorf (1966, 1968, 1978) applies and there exists a b_0 such that if $b > b_0$ there is a solution $L_s(b)$ such that

$$l_f[b, (a/b)^{1/3}, L_s(b)] = (1-a)\pi .$$

Thus, the solution of the restricted problem with these initial conditions is a symmetric periodic solution.

6. Stability of the Nearly Circular Orbits

By introducing the coordinates used by Poincaré (1892) we can easily establish that the nearly circular orbits found in Section 4 satisfy the 'twist condition' of Birkhoff, Arnold and Moser and so are of general elliptic type. Thus, a theorem of Birkhoff (1927) gives additional periodic solutions of long period which encircle these nearly circular orbits, and the theorems of Arnold (1961) and Moser (1962) establish the existence of invariant tori which are filled with quasi-periodic solutions which enclose the nearly circular orbits in an energy surface. The theorems of Arnold and Moser prove that these nearly circular orbits are stable.

The transformation from polar coordinates to Delaunay's elements given in the last section is singular at the circular orbits of the Kepler problem and so these elements do not constitute a valid coordinate system in a neighborhood of the circular orbits. This singularity is due to the geometric fact that the argument of the perihelion is not defined for circular orbits. Poincaré suggested another symplectic transformation which is again singular at the circular orbits, but such that the composition of the two transformations is not singular at the circular orbits. Thus, these new coordinates of Poincaré are valid in a neighborhood of the circular orbits.

There are two cases (1) the direct circular orbits and (2) the retrograde circular orbits and a corresponding set of coordinates for each case. We shall only consider the direct orbits since the other case is similar. Consider the symplectic change of variables from Delaunay's elements (l, g, L, G) to Poincaré variables (Q_1, Q_2, P_1, P_2) generated from the function

$$W_2 = P_1(l+g) - \tfrac{1}{2}P_2^2 \tan g\,. \tag{6.1}$$

The Hamiltonian (5.6) becomes

$$H = -P_1 + \tfrac{1}{2}(Q_2^2 + P_2^2) - \mu^3/2P_1^2 + O(\mu^5)\,, \tag{6.2}$$

and the equations of motion are

$$\begin{aligned} \dot{Q}_1 &= -1 + \mu^3/P_1^3 + O(\mu^5)\,, & \dot{Q}_2 &= P_2 + O(\mu^5)\,, \\ \dot{P}_1 &= O(\mu^5)\,, & \dot{P}_2 &= -Q_2 + O(\mu^5)\,, \end{aligned} \tag{6.3}$$

Q_1 is an angular variable and so defined modulo 2π, P_1 is a radial coordinate and (Q_2, P_2) are rectilinear coordinates. It can be shown that the composition map from $(\imath, \theta, R, \Theta)$ to (Q_1, Q_2, P_1, P_2) is an analytic, symplectic transformation in a neighborhood of the direct circular orbits of the Kepler problem (see Szebehely, 1967 or Schmidt, 1970). For a bounded time interval the solutions of (6.3) with initial conditions $(Q_{10}, Q_{20}, P_{10}, P_{20})$ at $t = 0$ are

$$\begin{aligned} Q_1 &= Q_{10} + t(1 + \mu^3/P_{10}^3) + O(\mu^5)\,, & Q_2 &= Q_{20}\cos t + P_{20}\sin t + O(\mu^5)\,, \\ P_1 &= P_{10} + O(\mu^5)\,, & P_2 &= -Q_{20}\sin t + P_{20}\cos t + O(\mu^5)\,. \end{aligned} \tag{6.4}$$

In order to apply the theorems of Birkhoff, Arnold and Moser we must compute the section (or Poincaré) map in an energy level. If we neglect the $O(\mu^5)$ terms in

(6.4), we note that for any P_{10} the solution which starts with $Q_{20} = P_{20} = 0$ is periodic of period $2\pi/(1-\mu^3/P_{10}^3)$ since the angular coordinate Q_1 is increased by 2π and all the other coordinates remain fixed. These are the direct circular orbits to this order of approximation. Let us compute the approximate section map in an energy level. The hypersurface defined by $\dot{Q}_1 = 0$ is transversal to the flow since $\dot{Q}_1 < 0$ for small μ. From the first equation in (6.4) we compute that the time $T = T(\mu, Q_2, P_1, P_2)$ of first return to this section is

$$T = 2\pi/(1+\mu^3 P_{10}^3) + O(\mu^5)$$
$$= 2\pi(1-\mu^3 P_{10}^3) + O(\mu^5) . \tag{6.5}$$

This is the time for Q_1 to decrease by 2π. Thus, using Q_2, P_1, P_2 as coordinates in this section we see that the section map takes (Q_{20}, P_{10}, P_{20}) to (Q_2, P_1, P_2) where

$$P_1 = P_{10} + O(\mu^5) ,$$
$$Q_2 = Q_{20} + 2\pi\mu^3 P_{10}^3 P_{20} + O(\mu^5) , \tag{6.6}$$
$$P_2 = P_{20} - 2\pi\mu^3 P_{10}^3 Q_{20} + O(\mu^5) .$$

Now fix the Hamiltonian by taking $H = 1$ and use this relation to eliminate P_1. Thus we take Q_2, P_2 as symplectic coordinates in the restriction of the cross-section $Q_1 = 0$ to the energy level $H = 1$. Thus, $P_{10} = -1 + \frac{1}{2}(Q_{20}^2 + P_{20}^2) + O(\mu^3)$ and so the section map in the energy level is $(Q_{20}, P_{20}) \to (Q_2, P_2)$ where

$$Q_2 = Q_{20} + 2\pi\mu^3 P_{20}[-1 + \tfrac{1}{2}(Q_{20}^2 + P_{20}^2)]^3 + O(\mu^5) ,$$
$$P_2 = P_{20} - 2\pi\mu^3 Q_{20}[-1 + \tfrac{1}{2}(Q_{20}^2 + P_{20}^2)]^3 + O(\mu^5) . \tag{6.7}$$

In order to give a second proof of the existence of the direct orbits established in Section 4, we apply the implicit function theorem to the equations

$$(2\pi\mu^3)^{-1}(Q_2 - Q_{20}) = P_{20}[1 + \tfrac{1}{2}(Q_{20}^2 + P_{20}^2)] + O(\mu^2) = 0 ,$$
$$(2\pi\mu^3)^{-1}(P_2 - P_{20}) = -Q_{20}[1 + \tfrac{1}{2}(Q_{20}^2 + P_{20}^2)] + O(\mu^2) = 0 . \tag{6.8}$$

When $\mu = 0$ the equations in (6.8) have a solution $Q_{20} = P_{20} = 0$ and the Jacobian of these equations with respect to Q_{20}, P_{20} is clearly non-zero when $\mu = Q_{20} = P_{20} = 0$. Thus, there exists a function $\tilde{Q}_2(\mu), \tilde{P}_2(\mu)$ for small μ which satisfy these equations and so are fixed points of the section map. Thus there exists nearly circular periodic solutions of the restricted problem.

If we shift this fixed point to the origin by changing variables by

$$Q_{20} \to Q_{20} - \tilde{Q}_2(\mu) \qquad Q_2 \to Q_2 - \tilde{Q}_2(\mu) ,$$
$$P_{20} \to P_{20} - \tilde{P}_2(\mu) , \qquad P_2 \to P_{20} - \tilde{P}_2(\mu) ,$$

the form of the section map (6.7) does not change except that now the origin is a fixed point for all small μ. Since the origin is always a fixed point we can introduce polar coordinates $I_0 = \frac{1}{2}(Q_{20}^2 + P_{20}^2)$, $I = \frac{1}{2}(Q_2^2 + P_2^2)$, $\theta_0 = \tan^{-1}(P_{20}/Q_{20})$, $\Theta =$

$\tan^{-1}(P_2/Q_2)$ and in these coordinates the section map (6.7) takes (I_0, θ_0) to (I, θ) where

$$I = I_0 = O(\mu^5)\,,$$
$$\Theta = \theta_0 - 2\pi\mu^3(-1 + I_0)^3 + O(\mu^5)\,. \tag{6.9}$$

Remark. Unless the origin is a fixed point of (6.7) this change of coordinates may introduce terms like μ^5/I in the $O(\mu^5)$ terms of (6.9). By first shifting the fixed point to the origin insures the $O(\mu^5)$ are uniform for all θ_0 and I_0 small.

Now that the section map has been placed in the form (6.9) we see that the celebrated 'twist condition' of Birkhoff (1927), Arnold (1961) and Moser (1962) holds for this map. Thus, by Birkhoff's theorem there exists an infinite number of periodic points of (6.9) which cluster on $I = 0$ and these give rise to very long periodic solutions of the restricted problem which cluster on the nearly circular orbits. Also the theorems of Arnold and Moser imply that (6.9) admits an infinite number of invariant curves which encircle $I = 0$ and cluster on $I = 0$. Thus, the nearly circular orbits are stable.

References

Arenstorf, R.: 1966, *J. Reine Angew. Math.* **221**, 113.
Arenstorf, R.: 1968, *J. Diff. Eqs.* **4**, 2, 202.
Arenstorf, R.: 1978, *Celestial Mech.* **17**, 331.
Arenstorf, R. and Boseman, R.: 1977, *Celestial Mech.* **16**, 179.
Arnold, V. A.: 1961, *Izv. Akad. Nauk. SSSR Ser. Math.* **25**, 21.
Birkhoff, G. D.: 1927, *Dynamical Systems*, Amer. Math. Soc., New York.
Coddington, E. A. and Levinson, N.: 1955, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York.
Hadjidemetriou, J. D.: 1975, *Celestial Mech.* **12**, 155.
Hartman, P.: 1964, *Ordinary Differential Equations*, John Wiley, New York.
Hale, J. K.: 1969, *Ordinary Differential Equations*, Wiley–Interscience, New York.
Meyer, K. R.: 1980, *J. Diff. Eqs.* (to appear).
Meyer, K. R. and Schmidt, D. S.: 1971, *Celestial Mech.* **4**, 99.
Moser, J.: 1962, *Nachr. Akad. Wiss. Göttengen Math. Phys.* K1, **IIa**, 1, 8.
Moulton, F. R.: 1912, *Trans. Amer. Math. Soc.* **13**, 96.
Poincaré, H.: 1892, *Les Methodes Nouvelles de la Mécaniques Céleste*, Gauthier-Villars, Paris.
Siegel, C.: 1956, *Vorlesungen über Himmelsmechanik*, Springer-Verlag, Berlin.
Schmidt, D. S.: 1970, Families of Periodic Orbits in the Restricted Problem of Three Bodies Connecting Families of Direct and Retrograde Orbits (Ph.D. Thesis), Univ. of Minnesota.
Smale, S.: 1970, *Invent. Math.* **II**, 45.
Szebehely, V.: 1967, *Theory of Orbits*, Academic Press, New York.