PERIODIC ORBITS NEAR $\mathscr{L}_4$ FOR MASS RATIOS NEAR THE CRITICAL MASS RATIO OF ROUTH

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Abstract. The Hamiltonian for orbits near $\mathscr{L}_4$ and mass ratios near μ_1 is brought into a normal form. A theorem shows that two coefficients in this expansion predict the behavior of the periodic orbits.

1. Introduction

Buchanan (1941) proved that there exist two families of periodic orbits which emanate from the Lagrange triangular point $\mathscr{L}_4$ in the restricted problem when the mass ratio parameter μ is equal to the Routh critical mass ratio μ_1. Several authors, Pedersen (1933, 1939), Deprit (1968), Palmore (1967), have investigated the behavior of these periodic solutions for $\mu > \mu_1$ either by numerical integration or approximate series expansions. These investigations have led to the conjecture that the two families detach as a unit from the equilibrium point and recede as μ increases from μ_1. We shall present a mathematical demonstration that this conjecture is correct. Moreover we compute the characteristic multipliers with sufficient accuracy to state that these periodic orbits are all of elliptic type in a sufficiently small neighborhood of the origin for μ sufficiently near μ_1 regardless of conjectures.

The method of proof is the continuation method of Poincaré. The essence of the argument is summarized in the perturbation lemma given in the next section. This perturbation lemma is applied to a general Hamiltonian system near a degenerate equilibrium to yield sufficient conditions for the existence of periodic orbits. These results are summarized in the theorem of Section 3. We then check the hypothesis of this general theorem for the Hamiltonian of the restricted problem at $\mathscr{L}_4$.

The authors would like to thank Dr J. Palmore for first calling their attention to this interesting problem.

2. The Perturbation Lemma

In this section we present a lemma which gives sufficient conditions for the existence of periodic solutions of a differential equation which admits an integral in a critical case. A standard theorem of Poincaré's continuation method states that if all but two

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of the characteristic multipliers of a periodic solution are not one then the periodic solution persists under a small perturbation provided both the perturbed and unperturbed systems are autonomous and admit an integral. (see Poincaré, 1892 or Siegel, 1956). This is the non critical case. The lemma given below considers the case when to the first approximation all periodic solutions have all characteristic multipliers equal to one. This is the critical case. In the critical case it is necessary to place further restrictions on the perturbation in order to insure the existence of some periodic solutions after the perturbation. The lemma given below is a natural extension of the perturbation theorems given in Coddington and Levinson (1955) chapter 14 to the case when the system admits an integral. Although this lemma is found implicitly in Lewis (1956) and Hale (1969) we shall present a proof here for completeness.

Consider the system

$$\dot{\xi} = A\xi + \varepsilon f(\xi, \varepsilon) \tag{1}$$

where $\xi \in R^n$, $\varepsilon \in R$, A is an $n \times n$ constant matrix such that $\exp(AT) = I$ for some $T > 0$ and f is an n-vector valued analytic function defined in a neighborhood of the origin in R^{n+1}. Since $\exp(AT) = I$ all solutions of (1) when $\varepsilon = 0$ are T-periodic. Assume that (1) admits an integral $I(\xi, \varepsilon) = \xi^T S\xi + \varepsilon H(\xi, \varepsilon)$ which is analytic in a neighborhood of the origin in R^{n+1}. Here S is an $n \times n$ constant symmetric matrix.

Let β be a real parameter and define

$$B(\beta, \xi) = \beta A\xi + \int_0^T e^{-As} f(e^{As}\xi, 0)\,\mathrm{d}s. \tag{2}$$

LEMMA: *If there exists analytic functions $\xi(\alpha)$, $\beta(\alpha)$ where α is a real parameter and $\xi(\alpha) \in R^n$ and $\beta(\alpha) \in R$ for $|\alpha| < \alpha_0$ such that*

$$B(\beta(\alpha), \xi(\alpha)) \equiv 0 \tag{3}$$

$$S\xi(\alpha) \neq 0 \tag{4}$$

$$\operatorname{rank}\left(\frac{\partial B}{\partial \beta}, \frac{\partial B}{\partial \xi}\right)(\beta(\alpha), \xi(\alpha)) = n - 1 \tag{5}$$

for $|\alpha| < \alpha_0$ then there exists an analytic two parameter family of periodic solutions of (1), *denoted by $\varphi(t, \alpha, \varepsilon)$, such that*

$$\varphi(t, \alpha, \varepsilon) \text{ is defined for all } t\text{; all } \varepsilon,\ |\varepsilon| < \varepsilon_0 \text{ and all } \alpha,\ |\alpha| < \tfrac{1}{2}\alpha_0 \tag{6}$$

$$\varphi(t, \alpha, 0) = (\exp At)\,\xi(\alpha) \tag{7}$$

$$\text{The period of } \varphi(t, \alpha, \varepsilon) \text{ is } T + \varepsilon\beta(\alpha) + 0(\varepsilon^2). \tag{8}$$

The characteristic multipliers of $\varphi(t, \alpha, \varepsilon)$ are $1 + \varepsilon\mu_i(\alpha) + 0(\varepsilon^2)$

$$\text{where } \mu_i(\alpha) \text{ are the eigenvalues of } \frac{\partial B}{\partial \xi}(\beta(\alpha), \xi(\alpha)). \tag{9}$$

PROOF: Let $\psi(t, \xi, \varepsilon)$ be the solution of (1) which satisfies $\psi(0, \xi, \varepsilon)=\xi$. In order to find periodic solutions of (1) it is enough to solve the equations $\psi(\tilde{T}, \xi, \varepsilon)=\xi$ for some $\tilde{T}\neq 0$, ξ and ε. By the variation of parameters formula

$$\psi(t, \xi, \varepsilon) = e^{At}\xi + \varepsilon \int_0^t e^{A(t-s)} f(\psi(s, \xi, \varepsilon), \varepsilon)\, \mathrm{d}s. \tag{10}$$

We introduce a new parameter β and compute

$$\psi(T + \varepsilon\beta, \xi, \varepsilon) = \xi + \varepsilon K(\beta, \xi, \varepsilon) \tag{11}$$

where

$$K(\beta, \xi, \varepsilon) = B(\beta, \xi) + 0(\varepsilon) \tag{12}$$

thus we must solve $K(\beta, \xi, \varepsilon)=0$ and in view of assumptions (3) and (5) we can solve $n-1$ of these equations by the implicit function theorem. Thus we must show these equations are dependent.

Since $I=\xi^T S\xi+\varepsilon H(\xi, \varepsilon)$ is an integral $\xi^T S\xi+\varepsilon H(\xi, \varepsilon)=(\xi+\varepsilon K)^T S(\xi+\varepsilon K)+\varepsilon H(\xi+\varepsilon K, \varepsilon)$ which rearranges to

$$\{2\xi^T S + 0(\varepsilon)\}\, K(\beta, \xi, \varepsilon) \equiv 0. \tag{13}$$

For a moment let α be fixed and choose coordinates so that $\xi^T S=(1, 0, \ldots, 0)$. Then we see by (13) that the first equation of the set $B(\beta, \xi)=0$ must be identically zero. Thus we may apply the implicit function theorem to solve the last $n-1$ equations in the set $K(\beta, \xi\ \varepsilon)=0$. But by (13) if the last $n-1$ equations of $K(\beta, \xi, \varepsilon)=0$ are satisfied then so must the first.

The characteristic multipliers of these periodic solutions are the eigenvalues of $\partial\psi/\partial\xi=I+\varepsilon\,\partial B/\partial\xi+0(\varepsilon^2)$ and so (9) follows.

3. The Main Theorem

In this section the lemma of the previous section is applied to a system of equations which has many of the properties of the restricted problem for mass ratio parameter μ near the Routh critical value μ_1. This theorem will be applied to the restricted problem in a subsequent section after the equations have been normalized.

Below we shall consider a real analytic system which has been transformed into a complex system in order to simplify the computations.

Recently Roels and Louterman (1970) have found the canonical form for a Hamiltonian matrix with repeated eigen values under a symplectic change of variables. For a 4×4 Hamiltonian matrix with repeated eigenvalues $i\omega$ and $-i\omega$, $\omega>0$, the canonical form is either diagonal or of the form

$$B_0 = \begin{bmatrix} \omega i & 1 & 0 & 0 \\ 0 & \omega i & 0 & 0 \\ 0 & 0 & -\omega i & 0 \\ 0 & 0 & -1 & -\omega i \end{bmatrix}. \tag{1}$$

The symplectic transformation to the new complex coordinates y_1, y_2, y_3, y_4 can always be done in such a way that the reality condition is

$$\bar{y}_1 = -cy_4$$
$$\bar{y}_2 = cy_3$$

where c is a real number. We will assume $c>0$ as the case $c<0$ is similar.

If we consider a Hamiltonian matrix $B(v)=B_0+vB_1+0(v^2)$ where v is a real parameter and $B_1=(b_{ij})$ then by the reality conditions $b_{21}=\bar{b}_{21}=-b_{34}=-\bar{b}_{34}=b$. The eigen values of $B(v)$ are $+i\omega+\sqrt{vb}+0(v)$. Thus if $b\neq 0$ and v is small the eigen values of $B(v)$ are pure imaginary for $vb<0$ and complex with non zero real part for $vb>0$. Henceforth we shall assume that $b>0$. This does not lead to a loss in generality since if $b<0$ one need only change v to $-v$.

Consider the Hamiltonian system of equations

$$\dot{y} = B(v)\,y + G(y, v) = J\nabla H \tag{2}$$

with Hamiltonian H, where y is a 4-vector and G is an analytic 4-vector valued function of y and the scalar v which is defined in a neighborhood of the origin in C^5. Assume that G has a convergent power series expansion in the y variables alone which starts with terms of degree at least two. The Hamiltonian function is therefore of the form

$$H = i\omega(y_1y_3 + y_2y_4) + y_2y_3 + \sum_{n=2}^{\infty} H_n(y, v) \tag{3}$$

where H_n is a homogeneous polynomial of degree n in the y variables and $H_2(y, 0)=0$.

Although for $v=0$ the eigenvalues of B_0 are not simple, the normalization procedure of Birkhoff (see Section 4) can still be used to eliminate all third order and most of the fourth order terms in (3) when $v=0$. This holds true for small $v\neq 0$. Thus we can assume H to be of the form

$$\begin{aligned} H = i\omega(y_1y_3 + y_2y_4) &+ y_2y_3 + H_2(y, v) \\ &+ a_1y_1^2y_3^2 + a_2y_1^2y_3y_4 + a_3y_1^2y_4^2 \\ &+ \sum_{n=5}^{\infty} H_n(y, v) \end{aligned} \tag{4}$$

with $H_2(y, 0)=0$. (Let $g=2a_3(0)$.)

The main theorem of this section is

THEOREM: (A) *If $g<0$ then there exists a neighborhood N of the origin in y-space, a $v_0>0$, an $h_0>0$ and a two parameter family of periodic solutions of* (2), *denoted by $p(t, v, h)$, which lie in N for all $|v|<v_0$, $|h|<h_0$, $h\neq 0$ when $v\leqslant 0$. The parameter h may be taken as the value of the Hamiltonian. The function p is analytic in all its arguments in its domain of definition. (Note that when $v\leqslant 0$ the domain of h does not contain* 0.*) For all values of v and h under consideration the periodic solution $p(t, v, h)$ is elliptic, i.e. two characteristic multipliers are not ± 1 and have unit modulus. For $\bar{v}$ fixed and $-v_0<\bar{v}\leqslant 0$, $p(t, \bar{v}, h)\to 0$ as $h\to 0^{\pm}$ and the frequence tends to $\omega\pm\sqrt{(-\bar{v}b)}+0(\bar{v})$.*

For $\bar{v}$ fixed and $0 < \bar{v} \leqslant v_0$ the one parameter family of periodic solutions $p(t, \bar{v}, h)$ does not contain the origin in y-space. (B) *If $g > 0$ then there exists a neighborhood N of the origin in y space, a $v_0 > 0$, an $h_0 > 0$ and an analytic two parameter family of periodic solutions of* (2), *denoted by $p(t, v, h)$ defined for $-v_0 < v < 0$ and $|h| < h_0$. In this case h cannot be taken as the value of the Hamiltonian and must be considered as just another parameter. For fixed $\bar{v}$, $-v_0 < \bar{v} < 0$, $p(t, \bar{v}, h) \to 0^-$ as $h \to +h_0$ and as $h \to -h_0$ and the frequence of $p(t, \bar{v}, h)$ tends to $\omega \pm \sqrt{(-\bar{v}b)} + 0(\bar{v})$. There exist an $h_1(\bar{v})$ and $h_2(\bar{v})$ such that $-h_0 < h_1(\bar{v}) < h_2(\bar{v}) < h_0$ and $p(t, \bar{v}, h)$ is elliptic for $-h_0 < h < h_1(\bar{v})$, $h_2(\bar{v}) < h < h_0$ and $p(t, \bar{v}, h)$ is hyperbolic for $h_1(\bar{v}) < h < h_2(\bar{v})$. As $v \to 0$, $p(t, v, h) \to 0$ for all h, $|h| < h_0$.*

REMARKS: Additional formulas for the dependence of the frequence and the multipliers on the parameters can be gleaned from the proof. For $v < 0$ and h small the family $p(t, v, h)$ is just the two families ($h < 0$ and $h > 0$) given by the well known Liapunov center theorem. Thus in case (A) the two families persist for $v = 0$ and as a unit detach from the origin as v becomes positive. In case (B) the two Liapunov families are globally connected for $v < 0$ and collapse into the origin as $v \to 0^-$.

PROOF: In order to apply the lemma of the previous section we must introduce a scale factor ε as a small parameter. Let

$$y_1 \to \varepsilon y_1, \qquad y_2 \to \varepsilon^2 y_2, \qquad y_3 \to \varepsilon^2 y_3, \qquad y_4 \to \varepsilon y_4, \qquad v \to \varepsilon^2 v.$$

This is a simplectic transformation with multiplier ε^3. Therefore from (4) the new Hamiltonian becomes

$$K = \varepsilon^{-3} H = i\omega(y_1 y_3 + y_2 y_4) + \varepsilon(y_2 y_3 + vby_1 y_4 + \tfrac{1}{2} g y_1^2 y_4^2) + 0(\varepsilon^2). \tag{5}$$

The reality condition implies that b and $g = 2a_3(0)$ are real. The system (2) becomes

$$y = J\nabla K = Cy + \varepsilon D(y, v) + 0(\varepsilon^2)$$

where $C = \mathrm{diag}(\omega i, \omega i, -\omega i, -\omega i)$ and

$$\begin{aligned} D_1(y, v) &= y_2 \\ D_2(y, v) &= vby_1 + gy_1^2 y_4 \\ D_3(y, v) &= -vby_4 - gy_1 y_4^2 \\ D_4(y, v) &= -y_3 . \end{aligned}$$

Thus we have reduced (2) to the form considered in the previous section and it remains to compute the bifurcation equations $B(\beta, y)$. They are

$$\begin{aligned} \omega B_1 &= \beta\omega^2 i y_1 + 2\pi y_2 \\ \omega B_2 &= \beta\omega^2 i y_2 + 2\pi(vby_1 + gy_1^2 y_4) \\ \omega B_3 &= -\beta\omega^2 i y_3 - 2\pi(vby_4 + gy_4^2 y_1) \\ \omega B_4 &= -\beta\omega^2 i y_4 - 2\pi y_3 . \end{aligned}$$

Solving $B_1 = 0$ for y_2 and substituting into $B_2 = 0$ yields $y_1(a\beta^2 + vb + gy_1 y_4) = 0$ where $a = \omega^4/(2\pi)^2$. A similar equation results from $B_3 = B_4 = 0$. By the reality conditions

$\bar{y}_1 = -cy_4$ and since $c > 0$ is assumed we introduce the real variable $r^2 = -y_1 y_4$. Thus we are led to the equation

$$a\beta^2 - gr^2 = -\nu b.$$

When $g > 0$ the graph of the above equation is as shown in Figure 1.

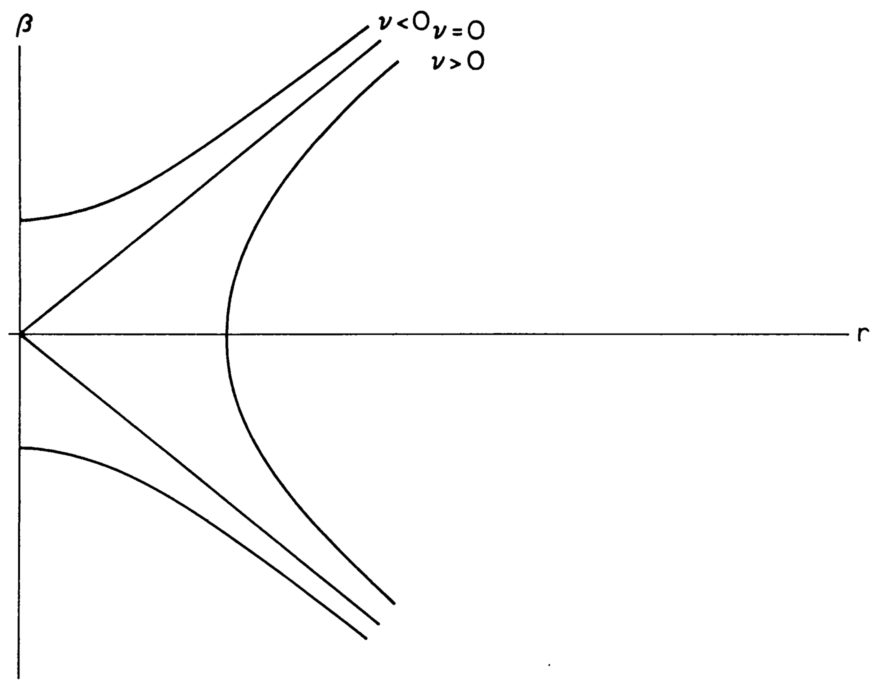

Fig. 1.

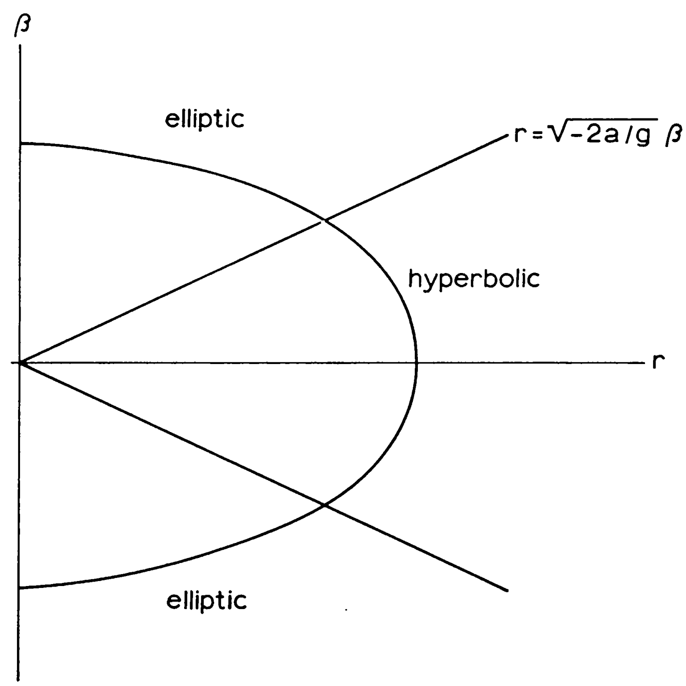

Fig. 2.

Thus for each fixed v the equation $B=0$ has a one parameter family of solutions. The qualitative features of this family are those described in part A of the above theorem. The value of the Hamiltonian (5) along this family is

$$K = h = -\frac{\omega^3}{\pi g}(vb\beta + a\beta^3) + 0(\varepsilon).$$

Since $\mathrm{d}h/\mathrm{d}\beta < 0$ for each family the value of the Hamiltonian can serve as parameter.

When $g<0$ the graph is as shown in Figure 2.

Again the qualitative features are described in part B of the above theorem.

It is left to show that the rank of $(\partial B/\partial\beta, \partial B/\partial y)$ is 3 along these families. In

$$\frac{\partial B}{\partial y} = \begin{pmatrix} i\omega\beta & 2\pi/\omega & 0 & 0 \\ \frac{2\pi}{\omega}(vb - 2gr^2) & i\omega\beta & 0 & \frac{2\pi}{\omega} g y_1^2 \\ -\frac{2\pi}{\omega} g y_4^2 & 0 & -i\omega\beta & -\frac{2\pi}{\omega}(vb - 2gr^2) \\ 0 & 0 & -\frac{2\pi}{\omega} & -i\omega\beta \end{pmatrix}$$

look at the submatrix which is formed by omitting the first column and the third row. Its determinant is $(2\pi/\omega)^3 g y_1^2$. Thus rank $(\partial B/\partial\beta, \partial B/\partial y) = 3$ unless $y_1 = 0$.

The nonzero eigenvalues of $\partial B/\partial y$ are $\pm 2\pi/\omega(-2gr^2 - 4a\beta^2)^{1/2}$. For $g>0$ the nontrivial characteristic multipliers are conjugate complex of unit modulus and $p(t, v, h)$ is elliptic. For $g<0$ and $-gr^2 < 2a\beta^2$, $p(t, v, h)$ is elliptic, if $-gr^2 > 2a\beta^2$, then $p(t, v, h)$ is hyperbolic.

4. Normalization of the Hamiltonian

Given is the Hamiltonian (Equation 3, Section 3)

$$H = i\omega(y_1 y_3 + y_2 y_4) + y_2 y_1 + \sum_{n=2}^{\infty} H_n(y, v) \tag{1}$$

with $H_2(y, 0) = 0$ and H_n is a homogeneous polynomial of degree n in the y variables, whose coefficients are functions of v.

First we want to show that for $v=0$ the terms of third order can be eliminated by a symplectic transformation to new variables $(\eta_1, \eta_2, \eta_3, \eta_4)$. Following Birkhoff (1927) this transformation is in first approximation

$$y_i = \eta_i + \frac{\partial K_3}{\partial \eta_{i+2}} \qquad y_{i+2} = \eta_{i+2} - \frac{\partial K_3}{\partial \eta_i} \quad i = 1, 2 \tag{2}$$

where $K_3 = K_3(\eta_1, \eta_2, \eta_3, \eta_4)$ is a homogeneous polynomial of degree 3 with terms $c_{ijk}\eta_i\eta_j\eta_k$ $1 \leqslant i \leqslant j \leqslant k \leqslant 4$. The transformation leaves second order terms unchanged and we try to determine the 20 coefficients c_{ijk} such that the third order terms disappear. This leads to

$$i\omega\left(\eta_3\frac{\partial K_3}{\partial\eta_3}-\eta_1\frac{\partial K_3}{\partial\eta_1}+\eta_4\frac{\partial K_3}{\partial\eta_4}-\eta_2\frac{\partial K_3}{\partial\eta_2}\right)+\eta_3\frac{\partial K_3}{\partial\eta_4}-\eta_2\frac{\partial K_3}{\partial\eta_1}+H_3=0\,.$$

Equating coefficients leads to a linear system of 20 equations for the c_{ijk}'s. Either by inspecting this system or by evaluating the corresponding determinant, which is $3^8\omega^{20}$ it is seen that the c_{ijk}'s are determined uniquely.

If $v\neq 0$ the presence of $H_2(y, v)$ will complicate the linear system of equations for the c_{ijk}'s, but the additional terms are of order $0(v)$. Thus for small $|v|$ the determinant will remain to be different from zero, and thus again we can find the c_{ijk}'s, such that in the transformed Hamiltonian all third order terms have disappeared.

The fourth order terms of the new Hamiltonian are treated accordingly by a transformation of the form (2), which uses now a homogeneous polynomial of degree 4. It turns out that all but 3 of the 40 fourth order terms in (1) can be eliminated. These are the terms containing $y_1^2y_3^3$, $y_1^2y_3y_4$ and $y_1^2y_4^2$, where in particular the coefficient of $y_1^2y_4^2$ is an invariant under this transformation.

5. Application to the Restricted Problem

The Hamiltonian of the restricted problem of three bodies at $\mathscr{L}_4$ starts with the following quadratic terms:

$$H_2=\tfrac{1}{2}(p_x^2+p_y^2)+yp_x-xp_y+\tfrac{1}{8}x^2-\frac{3\sqrt{3}}{4}(1-2\mu)\,xy-\tfrac{5}{8}y^2\,.$$

μ is the mass ratio and a critical value is $\mu_1=\tfrac{1}{2}\,(1-\tfrac{1}{9}\sqrt{69})$ when the corresponding Hamiltonian matrix has the canonical form (1) of Section 3. The repeated eigenvalues are $i/2\sqrt{2}$ and $-i/2\sqrt{2}$.

For mass ratios $\mu=\mu_1+v$ the corresponding matrix is of the form $B_0+vB_1+0(v^2)$. It has the eigenvalues

$$\pm\frac{i}{2}\sqrt{2}\pm\sqrt{vb}+0(v)\,. \tag{1}$$

On the other hand the characteristic equation belonging to the linearized system at $\mathscr{L}_4$ for any mass ratio is known to be

$$\lambda^4+\lambda^2+\tfrac{27}{4}\mu(1-\mu)=0\,.$$

It has the solutions

$$\begin{aligned}\lambda^2&=\tfrac{1}{2}(-1\pm\sqrt{1-27\,\mu(1-\mu)})\\&=\tfrac{1}{2}(-1\pm i\sqrt{3\sqrt{69}\,v})+0(v)\end{aligned}$$

comparing it with (1) we find

$$b=\tfrac{3}{2}\sqrt{69}\,.$$

In order to get the value for g we refer to the computations done by Deprit (1968). There the period of an orbit near $\mathscr{L}_4$ for $\mu=\mu_1$ was assumed to be $T=2\pi\sqrt{2}\,(1+v)$ i.e.

$$T - T_0 = 2\pi\sqrt{2}\,v.$$

The transformed Jacobian (Equations (130) and (124)) was determined to be

$$\begin{aligned}\Gamma = C - 3 &= \frac{216}{59}\,v^3 + \cdots \\ &= \frac{216}{59}\,\frac{(T - T_0)^3}{(2\pi\sqrt{2})^3} + \cdots.\end{aligned}$$

Now the value of the Hamiltonian (5) Section 3.

$$H = \varepsilon^2 K = \varepsilon^3\,\frac{i\sqrt{2}}{2}\,(y_1 y_3 + y_2 y_4) + 0(\varepsilon^4)$$

along the solution curves is

$$\begin{aligned}H &= -\,\varepsilon^3\,\frac{\beta}{2\pi}\,r^2 + \cdots \\ &= -\,\varepsilon^3\,\frac{1}{(2\pi)^3}\,\frac{1}{4g}\,(T - T_0)^3 + \cdots.\end{aligned}$$

Since $H=-\Gamma/2$ we can compare it with the result of Deprit and find

$$g = \frac{59\sqrt{2}}{216}.$$

Thus $g>0$ and the orbits near $\mathscr{L}_4$ for mass ratios near the critical mass ratio of Routh behave as described in part A of our theorem.

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Appendix

We would like to indicate briefly how the perturbation lemma of Section 2 can be applied to yield several known results in an easy and unified way.

First a degenerate form of the Liapunov center theorem. Consider a system whose Hamiltonian is

$$H(x_1, x_2, y_1, y_2) = k\omega I_1 + \omega I_2 + \tfrac{1}{2}(AI_1^2 + 2BI_1I_2 + CI_2^2) + K$$

where $I_i = (x_i^2 + y_i^2)/2$, $\omega > 0$, k is a non zero integer and K is an analytic function with a convergent power series expansion at the origin beginning with terms of degree at least 5. Here we see that the linearized system is two harmonic oscillators with one frequence a multiple of the other. The Liapunov center theorem yields a one parameter family of periodic orbits which emanate from the origin for $k \neq \pm 1$ and whose period tends to $2\pi/k\omega$ as the members of the family tend to the origin. But what about periodic orbits of periods near $2\pi/\omega$ i.e. the long period orbits? Introduce a scale factor $\varepsilon > 0$ by $I_i \to \varepsilon I_i$ and this system is now in a form that perturbation lemma requires. An easy calculation yields the bifurcation equations and one finds that there exists a one parameter family of periodic orbits whose period tends to $2\pi/\omega$ as they tend to the origin provided $B - kC \neq 0$. Moreover these orbits are of elliptic type. The Hamiltonian for the restricted problem near the Lagrange points $\mathscr{L}_4$ and $\mathscr{L}_5$ has been put into this canonical form by Deprit and Deprit-Bartholomé (1967). Using their results one easily checks that $B - kC \neq 0$ for $\mu = \mu_k$, $k = 4, 5, 6 \ldots$. This yields the existence of the long period family at $\mathscr{L}_4$ for $\mu = \mu_k$, $k = 4, 5, \ldots$. which is a result of Roels (1969). This result and method of proof was observed by the first author Dr J. Palmore and was announced in Palmore (1969). The canonical form at $\mathscr{L}_4$ for $\mu = \mu_3$ is slightly different but a similar analysis can be carried out in this case also.

In his investigation of Hill's orbits in the restricted problem Conley (1963) reduced the equations of motion in regularized coordinates to a special normal form (c.f. Equations (5), page 455 of Conley (1963)). One can again apply the lemma of section 2 to yield two one parameter families of periodic orbits about the primaries and thus give an alternate proof of Conley's theorem. We must point out that Conley obtains an analytic family whereas our proof does not give analyticity at the origin.

As our final example let us consider orbits at infinity in the restricted problem. The Hamiltonian of the restricted problem is

$$H = \tfrac{1}{2}(y_1^2 + y_2^2) - (x_1 y_2 - x_2 y_1) - (1 - \mu)/\varrho_1 - \mu/\varrho_2$$

where

$$\varrho_1^2 = (x_1 + \mu)^2 + x_2^2, \varrho_2^2 = (x_1 + \mu - 1)^2 + x_2^2.$$

Make the scale change

$$x_i \to \varepsilon^{-2/3} x_i, \qquad y_i \to \varepsilon^{1/3} y_i$$

which is a symplectic change of variables with multiplier $\varepsilon^{-1/3}$ so that the new

Hamiltonian $K=\varepsilon^{1/3}H$ becomes

$$K = -(x_1y_2 - x_2y_1) + \varepsilon\left\{\tfrac{1}{2}(y_1^2 + y_2^2) - \frac{1}{(x_1^2 + x_2^2)^{1/2}}\right\} + 0(\varepsilon).$$

Note that ε small means that we are near infinity in the original x_1, x_2 plane. The equations of motion derived from K are in the form required by the perturbation lemma. By calculating the bifurcation equations one finds that there exists two one parameter families of periodic orbits near infinity for all mass ratios $0 \leqslant \mu \leqslant 1$.