CONTRIBUTIONS TO DIFFERENTIAL EQUATIONS, VOL. III, NO. 4.

# A Remark on a Result of Lefschetz*

K. R. MEYER

In the recently published book on control theory by Aizerman and Gantmacher [1] it is pointed out that a condition for absolute stability as given by Lefschetz in a paper [2] published in this periodical holds vacuously. Lefschetz considers a system

$$
\begin{aligned}
x &= Ax + b\phi(\sigma) \\
\sigma &= c'x
\end{aligned}
\qquad (1)
$$

and a Liapunov function

$$V = x'Bx + \int_0^\sigma \phi(\sigma)\, d\sigma \qquad (2)$$

whose derivative along the trajectories of (1) is

$$
\begin{aligned}
&\text{(a)}\quad -\dot{V} = -(2Bx + \phi(\sigma)c)'(Ax + b\phi) \\
&\text{(b)}\quad -\dot{V} = x'Cx - 2(Bb + \tfrac{1}{2}A'c)'x\phi(\sigma) - c'b\phi^2(\sigma)
\end{aligned}
\qquad (3)
$$

In the above $x$, $b$, $c$, are real $n$ vectors with $x$ a function of the real variable $t$ and $\dot{x} = \dfrac{dx}{dt}$, $A$ is a real $n \times n$ stable matrix (i.e., all characteristic roots have negative real parts) and $C$ is an arbitrary real $n \times n$ positive definite symmetric matrix and $B$ is a positive definite symmetric matrix chosen so that $-C = A'B + BA$. The function $\phi(\sigma)$ is a continuous scalar function of the scalar $\sigma$ and satisfies $\sigma\phi(\sigma) > 0$, $\sigma \neq 0$; $\phi(0) = 0$ and such a function is called an *admissible function*. Thus $V$ is positive definite and tends to $\infty$ as $\|x\| \to \infty$. Now in order to make $-\dot{V}$ positive definite Lefschetz gives a condition that insures the quadratic form in $-\dot{V}$ is positive definite in the $n + 1$ variable $x$, $\phi(\sigma)$ where $\phi(\sigma)$ is considered to be independent of $x$. But the form of $-\dot{V}$ in (3a) shows that a common factor

* This research was supported in part by the Air Force Office of Scientific Research under Contract No. AF 49(638)-1242, in part by the National Aeronautics and Space Administration under Contract NASw-845, and in part by the Office of Naval Research Under Contract No. Nonr-3693-(00).

is $Ax + b\phi(\sigma)$ and if $\phi(\sigma)$ is allowed to vary independently of $x$ this factor set equal to zero has a nontrivial solution since there are $n$ equations and $n + 1$ unknowns. Thus the condition that makes $-\dot{V}$ positive definite in $x$ and $\phi(\sigma)$ cannot be satisfied.*

In order to avoid this difficulty Aizerman and Gantmacher (and others) use the simple device of adding and subtracting $\alpha\sigma\phi(\sigma)$, $\alpha > 0$ from $-\dot{V}$ and thus (3b) becomes

$$(4) \qquad -\dot{V} = \{x'Cx - 2(Bb + \tfrac{1}{2}A'c + \tfrac{1}{2}\alpha c')x\phi(\sigma) - c'b\phi(\sigma)^2\} + \alpha\sigma\phi(\sigma).$$

One can now make the term in the brackets positive definite in $x$ and $\phi(\sigma)$ without the above difficulties by using Lefschetz's condition

$$(5) \qquad -c'b > (Bb + \tfrac{1}{2}A'c + \tfrac{1}{2}\alpha c)'C^{-1}(Bb + \tfrac{1}{2}A'c + \tfrac{1}{2}\alpha c).$$

Thus (5) insures that $-\dot{V}$ is positive definite and hence (1) is absolutely stable (i.e., asymptotically stable in the large for all admissible functions).

An alternate procedure is as follows:

Assume (a) the system of equations $Ax + b\phi(c'x) = 0$ has only the solution $x = 0$ for all admissible $\phi$ and

(b) for some positive definite $C$,

$$c'b - (Bb + \tfrac{1}{2}A'c)'C^{-1}(Bb + \tfrac{1}{2}A'c) = 0.$$

The left hand side of the equality in (b) is just the determinant of the matrix of the quadratic form in (3b) and hence $-\dot{V}$ is positive semi-definite and is zero only on a one-dimensional subspace of the $n + 1$ dimensional space of $x$ and $\phi$. Now the system of equation $Ax + b\phi = 0$ ($\phi$ considered independent of $x$) is of rank $n$ since $A$ is nonsingular and so the set of solutions is a one-dimensional subspace of the space $x$ and $\phi$. Thus $\dot{V} = 0$ if and only if $Ax + b\phi = 0$, but condition (a) rules out the possibility of $Ax + b\phi(c'x)$ vanishing. Therefore (a) and (b) imply that $-\dot{V}$ is positive definite in the $n$ dimensional space of $x$ and hence imply absolute stability.

It is clear that the condition (a) states that the origin is the only critical point for the system (1). We shall now show that (a) is equivalent to the following condition

$$(\text{a}') \qquad c'A^{-1}b \geq 0.$$

Let (a′) hold and assume $Ax_0 + b\phi(c'x_0) = 0$ for some admissible function

---

* This observation was first made by E. N. Rozenvasser [3] in a note on I. G. Malkin's paper [4].

$\phi$ and some $x_0$. Since $\phi$ is an admissible function it is clear that $\phi(c'x_0) = \mu c'x_0$ for some $\mu > 0$ and thus $(A + \mu bc')x_0 = 0$. But

$$\det(A + \mu bc') = \{\det A\}\{\det(I + \mu A^{-1}bc')\}$$
$$= \{\det A\}\{1 + \mu c'A^{-1}b\}$$

and so (a′) implies that $A + \mu bc'$, $\mu > 0$, is nonsingular and so $x_0 = 0$.

If (a′) does not hold then $\phi(\sigma) = -(c'A^{-1}b)^{-1}\sigma$ is an admissible function for which (a) does not hold. Hence (a) and (a′) are equivalent.

The above alternate procedure is the result of an extended conversation with Professor Lefschetz and he has incorporated the above in his forthcoming monograph on control theory.

## References

[1] Aizerman M. A., and Gantmacher F. R., *Absolute Stability of Regulator Systems*, Acad. of Sci. USSR. Moscow, 1963.

[2] Lefschetz S., *Some mathematical considerations on nonlinear automatic controls*, Contrib. Diff. Eq. Vol. 1, 1960, p. 1.

[3] Rozenvasser E. N., *Remarks on a method for constructing Liapunov functions*, P.M.M., Vol. XXIV, 1960, p. 4.

[4] Malkin I. G., *On the theory of stability of control systems*, P.M.M., Vol. XV, 1951, p. 1.

*RIAS, Baltimore, Md., and University of Cincinnati, Cincinnati, Ohio.*

April 29, 1964.