On a System of Equations in Automatic Control Theory*

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I. Introduction

Letov [1] has introduced the study of a control system in which the equations of the control take into account the applied load. In particular he has taken an equation of Khokhlov that describes a loaded hydraulic servomotor and used this to describe the action of the automatic roll stabilization system in the *Queen Mary*. In this paper the system introduced by Letov will be examined with the aid of a lemma due to Yacubovich [2] as generalized by Kalman [3]. A rather complete answer can be given for the noncritical case as well as for some critical cases.

The system to be investigated is

$$\dot{v} = Av - b\mu \tag{1}$$

where v is the state vector and μ is the control. The control μ is governed by the equations

$$\begin{aligned} \dot{\mu} &= \psi(w)\phi(\sigma) \\ \sigma &= c'v - \rho\mu \\ w &= 1 - \theta\mu \operatorname{sgn} \sigma. \end{aligned} \tag{2}$$

In (1) and (2) v, b, c are n-vectors, $\mu, \sigma, w, \rho, \theta$ are scalars and A is an $n \times n$ matrix. The functions ψ, ϕ are scalar continuous functions such that (1) and (2) have unique solutions and satisfy the following conditions

$$\text{(3a)} \quad \sigma\phi(\sigma) > 0, \qquad \sigma \neq 0; \qquad \phi(0) = 0; \qquad \int_0^{\pm\infty} \phi(\sigma)\, d\sigma = +\infty$$

$$\text{(3b)} \quad \psi(w) > 0, \qquad w > 0; \qquad \psi(w) = 0, \qquad w \leqq 0$$

$[d\psi(w)/dw]$ exists and is continuous and $(d\psi/dw) \geqq 0$ when $w > 0$.

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Also the constant θ will be taken as nonnegative and $\rho \geqq 0$. The problem to be considered is to find conditions on the control parameters that insure asymptotic stability in the large for all such ϕ and ψ.

II. The Nonsingular Case

We will in this section consider the case where the matrix A has $2p$ simple imaginary characteristic roots and ℓ characteristic roots with negative real parts. Since we may take $p = 0$ we will be considering not only a critical case but also the noncritical case. We shall assume that (A, b) and (A, c') are completely controllable and completely observable respectively in order to apply the Kalman-Yacubovich lemma. A pair (A, b) is said to be completely controllable if $x' \exp \{At\} b \equiv 0$ for a finite interval of t implies that $x = 0$ and (A, c') is completely observable if and only if (A', c) is completely controllable. Following Lefschetz [4] we shall make the following change of coordinates $x = Av - b\mu$, $\sigma = c'v - \rho\mu$ and so (1) and (2) is equivalent to the following (4) provided $\rho \neq c'A^{-1}b$.

$$\begin{aligned} \dot{x} &= Ax - b\psi(w)\phi(\sigma) \\ \dot{\sigma} &= c'x - \rho\psi(w)\phi(\sigma) \\ w &= 1 - \theta\mu \operatorname{sgn} \sigma \\ \gamma\mu &= c'A^{-1}x - \sigma \end{aligned} \tag{4}$$

where $\gamma = \rho - c'A^{-1}b$. We shall assume without loss of generality that γ is positive. Let A be in the canonical form $A = \operatorname{diag}(K, \bar{K}, S)$ where $K = \operatorname{diag}(ik_1, \ldots, ik_p)$, the k's are distinct and positive, $\bar{K}$ is the conjugate of K, and S an $\ell \times \ell$ real stable matrix. The system (4) then reduces to

$$\begin{aligned} \dot{y} &= Ky - f\psi(w)\phi(\sigma) \\ \dot{\bar{y}} &= \bar{K}\bar{y} - \bar{f}\psi(w)\phi(\sigma) \\ \dot{z} &= Sz - d\psi(w)\phi(\sigma) \\ \dot{\sigma} &= g'y + \bar{g}'\bar{y} + e'z - \rho\psi(w)\phi(\sigma) \\ w &= 1 - \theta\mu \operatorname{sgn} \sigma \\ \gamma\mu &= g'K^{-1}y + \bar{g}'\bar{K}^{-1}\bar{y} + e'S^{-1}z - \sigma \end{aligned} \tag{5}$$

where y, f, g are p vectors and z, d, e are real ℓ vectors.

Consider the following Liapunov function for the system (5)

$$V = y'Q\bar{y} + z'Rz + \frac{\alpha\gamma}{2}\mu^2 + \beta\int_0^\sigma \psi(w)\phi(\sigma)\, d\sigma \tag{6}$$

where Q is a real positive definite diagonal matrix, R is a positive semi-definite symmetric matrix and $\alpha \geqq 0$, $\beta \geqq 0$, $\alpha + \beta > 0$. Now the derivative of (6) along the trajectories of (5) is

$$
\begin{aligned}
-\dot{V} = &-y'\{KQ + Q\bar{K}\}\bar{y} - z'\{S'R + RS\}z \\
&+ \{Q\bar{f} - \alpha K^{-1}g - \beta g\}'y\psi(w)\phi(\sigma) \\
&+ \{Qf - \alpha\bar{K}^{-1}\bar{g} - \beta\bar{g}\}'\bar{y}\psi(w)\phi(\sigma) \\
&+ 2\left\{Rd - \frac{\alpha}{2}S^{-1}e - \frac{\beta}{2}e\right\}' z\psi(w)\phi(\sigma) + \alpha\sigma\psi(w)\phi(\sigma) \\
&+ \beta\rho\{\psi(w)\phi(\sigma)\}^2 + \beta\theta\left\{\int_0^\sigma \frac{d}{dw}\psi(w)\phi(\sigma)\,d\sigma\right\}\psi(w)\phi(\sigma)\,\mathrm{sgn}\,\sigma.
\end{aligned} \tag{7}
$$

Now since Q is real and diagonal $KQ + Q\bar{K} = 0$. Assume that for some such Q

$$
\begin{aligned}
Q\bar{f} - \alpha K^{-1}g - \beta g &= 0 \\
Qf - \alpha\bar{K}^{-1}\bar{g} - \beta\bar{g} &= 0
\end{aligned} \tag{8}
$$

then an equivalent form for (7) is after completing the square

$$
\begin{aligned}
-\dot{V} = &\; z'(C - qq')z + (\sqrt{\tau}\,\psi(w)\phi(\sigma) + q'z)^2 \\
&+ \alpha\sigma\psi(w)\phi(\sigma) + \beta\theta\left[\int_0^\sigma \frac{d}{dw}\psi(w)\phi(\sigma)\,d\sigma\right]\psi(w)\phi(\sigma)\,\mathrm{sgn}\,\sigma
\end{aligned} \tag{9}
$$

where

$$-C = S'R + RS \tag{10a}$$

$$\tau = \beta\rho \tag{10b}$$

$$\sqrt{\tau}q = Rd - \frac{\alpha}{2}S'^{-1}e - \frac{\beta}{2}e. \tag{10c}$$

Now by the Kalman-Yacubovich lemma there exists a positive symmetric matrix R and a q satisfying $C - qq' = 0$ and (10a, b, c) if and only if

$$\beta\rho + \mathrm{Re}\,(\alpha e'S^{-1} + \beta e')S_{i\omega}{}^{-1}d \geqq 0 \tag{11}$$

for all real ω where $S_{i\omega} = (i\omega I - S)$.

We shall now show that (8) and (11) imply asymptotic stability in the large for the system (5). First we shall show that V is positive definite in y, z and σ. By the Kalman-Yacubovich lemma the set $\{z\colon z'Rz = 0\}$ is a linear space of unobservable states relative to $(S, \alpha e'S^{-1} + \beta e')$. If α or $\beta = 0$ then R is positive definite by the complete observability of (A, c')

and hence of (S, e'). If $\beta = 0$ then clearly V is positive definite since then $\alpha > 0$. Let $\alpha = 0$, and $\beta > 0$ then V is positive definite if $y = 0$, $z = 0$, $w \leqq 0$ implies that $\sigma = 0$, assume that $\sigma = \sigma_0 \neq 0$. Then $-\gamma\mu_0 = \sigma_0$ and $0 \geqq 1 - \theta\mu_0 \operatorname{sgn} \sigma_0$ or $0 \geqq 1 + \theta(\sigma_0/\gamma) \operatorname{sgn} \sigma_0$ which is a contradiction since γ and θ are positive. So V is positive definite if α or $\beta = 0$.

Now let $\alpha > 0$ and $\beta > 0$ and let us show that $z_0'Rz_0$ and $e'S^{-1}z_0$ cannot be both zero at the same time unless $z_0 = 0$. Assume the contrary. Then by the lemma z_0 is such that $(\alpha e'S^{-1} + \beta e')e^{St}z_0 \equiv 0$ for all t so by letting $t = 0$ this implies that $e'z_0 = 0$. By differentiating k times and letting $t = 0$ it follows that $e'S^kz_0 = 0$. But $[e', e'S, \ldots, e'S^{n-1}]$ are linearly independent vectors and hence $z_0 = 0$. Now that this fact has been established it follows easily by checking all possibilities that V is *positive definite.*

Now by the assumption on the divergence of the integral it follows that $V \to \infty$ as $|y|, |z|, |\sigma| \to \infty$ and so by a theorem of LaSalle [5] all solutions of (5) are bounded and tend to the largest invariant subset M of $E = \{(y, \bar{y}, z, \sigma) : \dot{V}(y, \bar{y}, z, \sigma) = 0\}$, as $t \to \infty$.

Case I. $\alpha \neq 0$. Let $P = (y_0, \bar{y}_0, z_0, \sigma_0) \neq 0$ be a point such that the solution of (5) starting at P for $t = 0$ remains in E for all t. Such a solution is a solution of the linear system obtained from (5) by letting $\psi(w)\phi(\sigma) = 0$, thus the solution is:

$$y = \{\exp Kt\}y_0, \qquad \bar{y} = \{\exp \bar{K}t\}\bar{y}_0, \qquad z = \{\exp St\}z_0$$

$$\sigma = \sigma_0 + g'K^{-1}\{\exp Kt\}y_0 + \bar{g}'\bar{K}^{-1}\{\exp \bar{K}t\}y_0 + e'S^{-1}\{\exp St\}z_0$$

$$\gamma\mu = -\sigma_0, \qquad w = 1 + \theta\gamma^{-1}\sigma_0 \operatorname{sgn} \sigma.$$

Now σ cannot be zero for a finite time interval since this would contradict the complete observability of (A, c') and thus $w \leqq 0$ and $\sigma_0 \neq 0$. But since S is stable there exists a T such that for all $t > T$, $|e'S^{-1} \exp \{St\}z_0| < (|\sigma_0|/4)$ and since $g'K^{-1} \exp \{Kt\}y_0 + \bar{g}'\bar{K}^{-1} \exp \{\bar{K}t\}\bar{y}_0$ is an almost periodic function with zero mean value there exists a $t^* > T$ such that it is less than $(|\sigma_0|/4)$ in absolute value for all t in a small interval around t^*. Therefore near t^*, $w = 1 + \theta\gamma^{-1}\sigma_0 \operatorname{sgn} \sigma_0 > 0$ which is a contradiction. Thus the largest invariant subset of E is $M = \{0\}$.

Case II.

$$\alpha = 0, \qquad -\dot{V} = (\sqrt{\tau}\, \psi(w)\phi(\sigma) + q'z)^2 + \beta\theta\left[\int_0^\sigma \frac{d}{dw}\, \psi(w)\phi(\sigma)\, d\sigma\right]\psi(w)\phi(\sigma) \operatorname{sgn} \sigma.$$

A solution remaining in E must be bounded for all t in $(-\infty, \infty)$ since such a solution must lie in a level surface of V and also for such a solution $\sqrt{\tau}\psi(w)\phi(\sigma) = -q'z(t)$. Since $\alpha = 0$ we may take $\beta = 1$ and thus the equation for z reduces to the linear equation $\dot{z} = (S + \rho^{-\frac{1}{2}}\, dq')z$ if $\rho \neq 0$ or to $\dot{z} = \{S - (q'S^m\, d)^{-1}\, dq'S^{m+1}\}z$ if $\rho = q'd = \cdots = q'S^{m-1}\, d = 0$ and $q'S^m\, d \neq 0$ (such an m exists by the Kalman-Yacubovich lemma). A solution of a linear equation that is bounded for all t must be the sum of exponentials with pure imaginary exponents thus $\psi(w)\phi(\sigma)$ is of the form $\psi(w)\phi(\sigma) = \sum_{-n}^{n} a_j \exp\{i\omega_j t\}$ where $a_j = \bar{a}_{-j}$, $\omega_j = -\omega_{-j}$, $\omega_0 = 0$. Using this form for $\psi(w)\phi(\sigma)$ in eq. (4) we can calculate $x(t)$ and $\sigma(t)$. Since such a solution must be bounded it follows that $a_0 = 0$ and $\omega_s \neq k_r$ for any s and r. Letting Σ' denote sum excluding $j = 0$ we have

$$x(t) = -\sum_{-n}^{n}{}' a_j(A_{i\omega_j}^{-1}b) \exp\{i\omega_j t\} + \sum_{-p}^{p}{}' v_j \exp\{ik_j t\} + v_0$$

$$\psi(w)\phi(\sigma) = \sum_{-n}^{n}{}' a_j \exp\{i\omega_j t\}$$

$$\sigma(t) = \sigma_0 - \sum_{-n}^{n}{}' a_j(i\omega_j)^{-1}(\rho + c'A_{i\omega_j}^{-1}b) \exp\{i\omega_j t\}$$

$$+ \sum_{-p}^{p}{}' c'v_j(ik_j)^{-1} \exp\{ik_j t\} + c'v_0$$

where $v_j = \bar{v}_{-j}$ are n vectors and σ_0 is a constant. We can assume we are not in Case I and so $\psi(w)\phi(\sigma) \not\equiv 0$ and since $\sigma(t)\psi(w(t))\phi(\sigma(t)) \geqq 0$ it follows that

$$\lim_{T\to\infty} \int_0^T \sigma(t)\psi(w(t))\phi(\sigma(t))\, dt = -\sum_{-n}^{n} a_j a_{-j}(i\omega_j)^{-1}(\rho + c'A_{i\omega_j}^{-1}b) > 0$$

We shall have a contradiction and thus prove our theorem once we establish the following:

LEMMA. *Let the linearized system obtained from* (1) *by placing* $\theta = 0$ *and* $\psi(1)\phi(\sigma) = \nu\sigma$ *be asympotically stable for all* $\nu > 0$. *Then if* $i\omega_0$ *is a characteristic root of* $S + \rho^{-\frac{1}{2}}\, dq'$ *if* $\rho \neq 0$ *or of* $S - (q'S^m\, d)^{-1}\, dq'S^{m+1}$ *if* $\rho = 0$ *and* m *is as above such that* $i\omega_0 \neq \pm ik_j$ *for any* j *then* $(i\omega_0)^{-1} \times (\rho + c'A_{i\omega_0}^{-1}b)$ *is a nonnegative real number.*

Proof. The characteristic equation of the linearized system is $|A_\lambda|\,\{\lambda + \nu(\rho + c'A_\lambda^{-1}b)\}$ and so $(i\omega_0)^{-1}(\rho + c'A_{i\omega}{}^{-1}b)$ can not be a negative real number since in this case there is a positive $\tilde{\nu}$ such that

$i\omega_0$ is a characteristic root of the matrix of the linearized system with $\psi(\omega)\phi(\sigma) = \tilde{\nu}\sigma$. The lemma then follows if we show $\text{Re}\,(\rho + c'A_{i\omega_0}^{-1}b) \equiv \text{Re}\,(\rho + e'S_{i\omega_0}^{-1}\,d) = 0$. The characteristic equation of $S + \rho^{-\frac{1}{2}}\,dq'$ is

$$|\lambda I - S - \rho^{-\frac{1}{2}}\,dq'| = |S_\lambda - \rho^{-\frac{1}{2}}\,dq'| = |S_\lambda|\,|I - \rho^{-\frac{1}{2}}\,dq'S_\lambda^{-1}|$$
$$= |S_\lambda|\,\{1 - \rho^{-\frac{1}{2}}q'S_\lambda^{-1}\,d\}$$

and so $\rho = \rho^{\frac{1}{2}}q'S_{i\omega_0}^{-1}\,d = d'RS_{i\omega_0}^{-1}\,d - \frac{1}{2}e'S_{i\omega_0}^{-1}\,d$. Now $qq' = -C = \bar{S}'_{i\omega_0}R + RS_{i\omega_0}$ and so $2\,\text{Re}\,d'RS_{i\omega_0}^{-1}\,d = |q'S_{i\omega_0}^{-1}\,d|^2 = \rho$. By combining we have $\text{Re}\,(\rho + e'S_{i\omega_0}^{-1}\,d) = 0$. A similar argument holds if $i\omega_0$ is a characteristic root of $S - (q'S^m\,d)^{-1}\,dq'S^{m+1}$.

Thus we have shown that (8) and (11) along with the added assumption about the linearized system imply asymptotic stability in the large.

We will now put (8) and (11) into an invariant form that is in a form that can be applied directly to (4) without reducing (4) to the Jordan form (5). Assume that there exists $\alpha \geqq 0$, $\beta \geqq 0$, $\alpha + \beta > 0$ such that

$$\beta\rho + \text{Re}\,(\alpha c'A^{-1} + \beta c')A_{i\omega}{}^{-1}b \geqq 0 \tag{13}$$

for all real ω such that $\omega \notin \{\pm k_1, \pm k_2, \ldots, \pm k_p\}$.

$$(\alpha c'A^{-1} + \beta c')A_{i\omega}{}^{-1}b = (\alpha g'K^{-1} + \beta g')K_{i\omega}{}^{-1}f + (\alpha\bar{g}'K^{-1} + \beta\bar{g}')\bar{K}_{i\omega}{}^{-1}f + (\alpha e'S^{-1} + \beta e')S_{i\omega}{}^{-1}d. \tag{14}$$

It follows from the fact that S is a stable matrix that the last term in (14) is bounded for all real ω and hence

$$\text{Re}\,\{(\alpha g'K^{-1} + \beta g')K_{i\omega}{}^{-1}f + (\alpha\bar{g}'\bar{K}^{-1} + \beta\bar{g}')\bar{K}_{i\omega}{}^{-1}f\} \geqq M > -\infty \tag{15}$$

but the functions in the brackets in (15) is of the form

$$\sum_1^p \left\{\frac{a_j + ib_j}{i\omega - ik_j} + \frac{a_j - ib_j}{+i\omega + ik_j}\right\}$$

and hence $b_j = 0$ or that $(\alpha g_j(ik_j)^{-1} + \beta g_j)f_j$ is real.

Now if we make the further assumption that $(\alpha g_j(ik_j)^{-1} + \beta g_j)f_j = h_j > 0$ we see that $Q = \text{diag}\,[(h_1/|f_1|^2), (h_2/|f_2|^2), \ldots, (h_p/|f_p|^2)]$ satisfies (8), and (13) reduces to (11). By noting that $\lambda A^{-1}A_\lambda^{-1} = A_\lambda^{-1} + A^{-1}$ we have:

THEOREM 1A. *Let A have $2p$ ($p \geqq 0$) simple imaginary characteristic roots and ℓ roots with negative real parts and (A, b) and (A, c') be completely controllable and completely observable respectively. Also let $\theta \geqq 0$ and*

$\rho \geqq 0$. *Then the system* (1) *and* (2) *is asymptotically stable in the large for all* $\psi(w)$ *and* $\phi(\sigma)$ *satisfying* 3a *and* 3b *provided*:

(*a*) $\gamma = \rho - c'A^{-1}b > 0$

(*b*) *There exists constants* $\alpha \geqq 0$, $\beta \geqq 0$, $\alpha + \beta > 0$ *such that* $\beta\rho + \operatorname{Re}(\alpha/\lambda) + \beta)c'A_\lambda^{-1}b \geqq 0$ *for all imaginary* λ *not equal to a pole and* $((\alpha/\lambda) + \beta)c'A_\lambda^{-1}b$ *has positive residues on the imaginary axis.*

(*c*) *if* $\alpha = 0$ *then the linearized system obtained from* (1) *by letting* $\theta = 0$ *and* $\psi(1)\phi(\sigma) = \nu\sigma$ *is asymptotically stable for all* $\nu > 0$.

Now we shall show that the conditions given in Theorem 1A are also necessary for the existence of a positive Liapunov function of the type quadratic form plus integral of the nonlinearities.

For the first part of the argument let us assume that the eq. (4) are in the real form

$$\begin{aligned} \dot{y} &= Ky - f\psi(w)\phi(\sigma) \\ \dot{z} &= Sz - d\psi(w)\phi(\sigma) \\ \dot{\sigma} &= g'y + e'z - \rho\psi(w)\phi(\sigma) \\ w &= 1 - \theta\mu \operatorname{sgn} \sigma \\ \gamma\mu &= g'K^{-1}y + e'S^{-1}z - \sigma \end{aligned} \tag{16}$$

where now y, f, g are real $2p$ vectors and

$$K = \operatorname{diag}\left\{\begin{pmatrix} 0 & k_1 \\ -k_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & k_p \\ -k_p & 0 \end{pmatrix}\right\}$$

By the same argument as in Popov [6] the most general Liapunov function of the type quadratic form plus integral of the nonlinearities is

$$V = y'B_1y + 2y'B_2z + z'B_3z + \frac{\alpha\gamma}{2}\mu^2 + \beta\int_0^\sigma \psi(w)\phi(\sigma)\,d\sigma. \tag{17}$$

We shall also use the ϵ-method used by Popov which consists of substituting for the variables in V or $\dot{V}$ a power of ϵ times the variable. The sign of V or $\dot{V}$ is then determined by the lowest degree term in ϵ. Thus in $\dot{V}$ let $y \to y$, $z \to \epsilon z$, and $\psi(w)\phi(\sigma) \to \epsilon^2\psi(w)\phi(\sigma)$ then the lowest degree is $y'(K'B_1 + B_1K)y$. Since the diagonal elements in $K'B_1 + B_1K$ cancel it follows that if $\dot{V}$ is to be negative semi-definite that $K'B_1 + B_1K = 0$ and this implies that B_1 must be diagonal. Now the next lowest degree term in $\dot{V}$ is $2y'(K'B_2 + B_2S)z$ and clearly for $\dot{V} \leqq 0$ we must have $K'B_2 + B_2S = 0$ or $B_2 = 0$.*

* See Gantmacher, *The Theory of Matrices*, Vol. I, Chelsea, New York, 1959, p. 220.

Now to say that the form for V for eq. (16) must be such that B_1 is diagonal and $B_2 = 0$ is equivalent to saying that the form of the Liapunov function (6) was the only one possible for the eq. (5). So let us consider now (6) and (7). By letting $y \to y$, $\bar{y} \to \bar{y}$, $z \to \epsilon^2 z$, $\sigma \to \epsilon\sigma$, $\psi(w)\phi(\sigma) \to \epsilon\psi(w)\phi(\sigma)$ the lowest degree term in (7) is $\{Q\bar{f} - \alpha K^{-1}g - \beta g\}'y\psi(w)\phi(\sigma) + \{Qf - \alpha K^{-1}\bar{g} - \beta\bar{g}\}'\bar{y}\psi(w)\phi(\sigma)$ so (8) must hold. By the necessity part of the Kalman-Yacubovich lemma (11) must hold and (8) and (11) imply part b of Theorem 1A.

Now by picking a $\psi(w)$ that is equal to a constant in some neighborhood of 1 and satisfying the conditions of 3b and $\phi(\sigma) = \nu\sigma$ we see that in some neighborhood of the origin the equations reduce to the linearized system and hence the condition (c) is necessary. For the same $\psi(w)$ the determinant of the matrix of the linearized system is $\nu\,|A|\,(\rho + c'A_\lambda^{-1}b)$ and since a determinant is the product of the characteristic roots (a) is also necessary. Thus:

THEOREM 1B. *If for the system as defined in Theorem* 1A *there exists a positive definite Liapunov function of the type quadratic form plus integral of the nonlinearities whose derivative is nonpositive and the system is asymptotically stable in the large then the conditions* (a), (b) *and* (c) *of theorem* 1A *are satisfied.*

III. The Singular Case

Let us return to the system (1) and assume that A has a simple characteristic root zero and the other $n - 1$ characteristic roots have negative real parts. It should be noted that we could assume that A also has $2p$ simple imaginary characteristics roots and use again the methods of the previous section but we shall not do this in order to avoid lengthy arguments. Again we assume that (A, b) and (A, c') are completely controllable and completely observable respectively. Let us assume that (1) and (2) are in the canonical form

$$(1')\qquad \begin{aligned} \dot{\tilde{v}} &= S\tilde{v} - d\mu \\ \dot{y} &= -f\mu \\ \dot{\mu} &= \psi(w)\phi(\sigma) \end{aligned}$$

$$(2')\qquad \begin{aligned} \sigma &= e'\tilde{v} + gy - \rho\mu \\ w &= 1 - \theta\mu \operatorname{sgn} \sigma \end{aligned}$$

where $\tilde{v}$, d, e are $(n-1)$ vectors, y, f, g are scalars and S is an $(n-1) \times (n-1)$ stable matrix. $v' = (\tilde{v}', y)$, $b' = (d', f)$, and $c' = (e', g)$. Now making the change of coordinates

$$z = \dot{\tilde{v}} = S\tilde{v} - d\mu$$
$$\sigma = e'\tilde{v} + gy - \rho\mu$$

the above system is equivalent to

$$\begin{aligned} \dot{z} &= Sz - d\psi(w)\phi(\sigma) \\ \dot{y} &= -f\mu \\ \dot{\sigma} &= e'z - gf\mu - \rho\psi(w)\phi(\sigma) \\ w &= 1 - \theta\mu \operatorname{sgn} \sigma \\ \gamma\mu &= (\rho - e'S^{-1}d)\mu = e'S^{-1}z + gy - \sigma \end{aligned} \tag{18}$$

provided $\gamma = \rho - e'S^{-1}d \neq 0$ and we can assume again without loss of generality that $\gamma > 0$. Consider the following Liapunov function for the above system

$$V = z'Bz + \frac{\alpha\gamma}{2}\mu^2 + \beta\int_0^\sigma \psi(w)\phi(\sigma)\,d\sigma \tag{19}$$

$$\begin{aligned} -\dot{V} = {} & -z'\{S'B + BS\}z + 2\left\{Bd - \tfrac{1}{2}(\alpha - \beta gf\gamma^{-1})S'^{-1}e - \frac{\beta}{2}e\right\}' z\psi(w)\phi(\sigma) \\ & + (\alpha - \beta gf\gamma^{-1})\sigma\psi(w)\phi(\sigma) + (\alpha - \beta gf\gamma^{-1})gy\psi(w)\phi(\sigma) \\ & + \beta\rho\psi(w)\phi(\sigma) + \beta\theta\left\{\int_0^\sigma \frac{d\psi(w)}{dw}\phi(\sigma)\,d\sigma\right\}\psi(w)\phi(\sigma)\operatorname{sgn}\sigma. \end{aligned} \tag{20}$$

Now if $gf > 0$ we may take $\alpha > 0$ and $\beta > 0$ such that $\alpha - \beta gf\gamma^{-1} = 0$ and complete the square as in the previous section to obtain

$$\begin{aligned} -\dot{V} = {} & z'\{C - qq'\}z + (\sqrt{\tau}\psi(w)\phi(\sigma) + q'z)^2 \\ & + \beta\left\{\int_0^\sigma \frac{d\psi(w)}{dw}\phi(\sigma)\,d\sigma\right\}\theta\psi(w)\phi(\sigma)\operatorname{sgn}\sigma \end{aligned}$$

where

$$-C = S'B + BS$$
$$\tau = \beta\rho$$
$$\sqrt{\tau}q = Bd - \frac{\beta}{2}e'.$$

Now by the Kalman-Yacubovich lemma there exists a B and a q such that $C - qq' = 0$ and satisfies the above if

$$\rho + \text{Re } e'S_{i\omega}^{-1}d \geqq 0 \tag{22}$$

for all real ω or what is equivalent if

$$\rho + \text{Re } c'A_{i\omega}^{-1}b \geqq 0$$

for all nonzero real ω.

Now by an argument similar to the one found in the previous section, we have

THEOREM 2A. *If A has a simple zero characteristic root and the other characteristic roots have negative real parts then* (1) *and* (2) *or* (18) *is asymptotically stable in the large for all $\phi(\sigma)$, $\psi(w)$ satisfying* (3a) *and* (3b) *provided*

(*a*) *(A, b) and (A, c') are completely controllable and completely observable.*

(*b*) $\theta \geqq 0$.

(*c*) *The residue of $c'A_\lambda^{-1}b$ at the origin is positive.*

(*d*) *$\rho + \text{Re } c'A_{i\omega}^{-1}b \geqq 0$ for all nonzero real ω.*

(*e*) *The linearized system obtained from* (18) *by letting $\theta = 0$ and $\psi(1)\phi(\sigma) = \nu\sigma$ is asymptotically stable for all $\nu > 0$.*

THEOREM 2B. *Let A be as in Theorem* 2A. *Then if there exists for the system* (1) *and* (2) *or* (18) *a positive definite Liapunov function of the type quadratic form plus integral of the nonlinearities whose derivative is nonpositive then the conditions* (*c*) *and* (*d*) *of Theorem* 2A *are satisfied.*

Remarks. It should be noted that when $\theta = 0$ the system (1) and (2) reduces to the indirect control system of the Lurie type. Yacubovich [7] has shown for the Lurie system that the matrix A can not have an imaginary characteristic root of multiplicity greater than two and a zero characteristic root of multiplicity greater than one. It can also be shown by the ϵ-method that you cannot have a positive definite Liapunov function of the type quadratic form plus integral of the nonlinearities whose derivative is nonpositive for the case when A has an imaginary characteristic root of multiplicity two. Thus the results given in this paper are as general as can be obtained by this particular type of Liapunov function.

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ERRATA

ON A SYSTEM OF EQUATIONS IN AUTOMATIC CONTROL THEORY

To Theorem 1A one must add the hypothesis that $\alpha \neq 0$ when $\theta \neq 0$. If $\alpha = 0$ when $\theta \neq 0$ then one can only conclude stability in the small.

If $\theta' \neq 0$ you can only conclude stability in the small in Theorem 1B.