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BIBLIOGRAPHIC NOTES ON GENERIC BIFURCATIONS IN HAMILTONIAN SYSTEMS

by

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ABSTRACT. A brief discussion is given of the literature on generic bifurcation in Hamiltonian systems.

About ten years ago I wrote a rather superficial survey of generic bifurcations of periodic solutions in Hamiltonian systems (don't see Meyer (1975)). Here I would like to define a more limited bifurcation problem and give a more complete bibliography for this problem. My reason for writing these notes are twofold. First, there is a burst of interest in multiparameter bifurcations and the problem I will describe is a two parameter problem with a wealth of interest phenomena. And secondly, since the references are in both the Astronomy and Mathematics journals often researchers only know half the literature.

As background I assume that the reader is reasonable conversant in the basic theory of Hamiltonian differential equations. Here and below all systems will be autonomous. Since Hamiltonian systems admit the Hamiltonian as an integral, the value of the Hamiltonian acts as a parameter in the study of periodic solutions. Therefore, a single Hamiltonian system usually displays many bifurcations of periodic solutions as the integral is varied. A lively discussion of these bifurcations can be found in Chapter 8 of Abraham and Marsden (1978). Here you will find a complete description of the main results of Buchner (1970), Meyer (1970), Meyer and Palmore (1970a), Robinson (1970, 1971) and Takens (1970). Most of the results on normal forms and their application to the bifurcations of periodic solution can be found in Bruno (1970a, 1970b, 1972).

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Several interesting results either were omitted from or have appeared since Abraham and Marsden (1978). Rimmer (1978, 1983) a studied the generic bifurcation of periodic points in the case when there is a Z_2 symmetry. Klingenberg and Takens (1972) carry over the theory to geodesic flows (see the earlier Abraham (1970) also). Markus and Meyer (1974) use these bifurcation results to prove the generically Hamiltonian systems are neither integrable nor ergodic. Markus and Meyer (1980) show that generically Hamiltonian systems have solenoids minimal sets and these are the limits of bifurcations of periodic orbits. Thus the theory of bifurcations in a single Hamiltonian is fairly well understood.

The theory of the bifurcations of periodic solutions of a Hamiltonian system which depends on one or more parameters is not so fully developed, but there are many interesting results. The number of cases that must be considered grows very rapidly as the number of parameters increase. The only problem that is nearly complete is the study of the bifurcations of the periodic solutions that emanate from an equilibrium when one parameter is varied.

Consider a Hamiltonian system depending on the parameter which has an equilibrium at the origin. The eigenvalues of the linearized system appear in negative pairs and so may be ordered, $\lambda_1, \ldots, \lambda_n, -\lambda_1, \ldots, -\lambda_n$. It is generic (codimension zero) that $\lambda_1, \ldots, \lambda_n$ are independent over the integers. In that case a classical theorem of Liapunov states that each pair of pure imaginary eigenvalue gives rise to a one parameter family of periodic solutions which emanate from the origin.

Codimension 1 bifurcations of these Liapunov families occur when there is one integer relation between the eigenvalues. In this case there is no loss in phenomena in assuming the system has only two degrees of freedom. The prototype is the restricted three body problem at the Lagrange triangular equilibrium point. The numerical experiments on the periodic solutions of the three body problem in Deprit and Henrard (1968, 1969, 1970) and Palmore (1969) give the conjectures from which a generic theory can be made. These papers contain a wealth of information that has not yet been digested in the mathematical literature.

The important cases are when the linearized system is like two harmonic oscilators with frequences, ω_1, ω_2 and the ratio of these frequencies is p/q where p and q are relatively prime integers.

Case 1, p/q=1/1: This case divides into two subcases depending on whether the Hamiltonian is (a) positive definite or (b) indefinite. Since subcase (b) occurs in the restricted problem it has a longer history. Buchanan (1941) showed that when this case occurs in the restricted problem the two Liapunov families persist. Palmore (1969) investigated this case as the parameter moves away from resonance and found that the two families persist and detach from the origin. In Meyer and Schmidt (1971) it is shown that this phenomena is one of the two generic cases that occur in the 1:1 resonance, indefinite case. In this paper the general theorem is applied to the restricted problem, but there is a minor error in the calculation of one of the coefficients of the normal form. Several of the formulas in the last section are off by a spurious factor of $\sqrt{2}$ and the last formula should be g = 59/216. The correct calculation of the coefficients in the normal form were carried out in Deprit and Henrard (1968) in different coordinates. The spurious factor came from the calculation of the change of coordinates in Meyer and Schmidt (1971). van der Meer (1982) redid the calculation of Deprit and Henrard (1968) and thus the spurious factor was found. Kummer (1978) made a major breakthrough in this problem when he gave a systematic way of developing the normal form using group theory. This paper contains a wealth of information which is still being digested today.

The case when the Hamiltonian is positive definite is more difficult. The only major contribution to this problem that I know of is Kummer (1976). This is the first paper to use group theory to discuss the normal forms. Kummer does not give the complete unfolding in this case.

Case 2, p > q > = 1: There are many subcases, but they are all discussed in Henrard (1970b, 1973) and Schmidt (1973). Liapunov families persist or detach from the origin or collapse on the origin. Additional families of periodic solutions bifurcate from the Liapunov families and sometimes make global connections between the two Liapunov families. Here again the prototype is the restricted problem and the numeric studies in Deprit and Henrard (1968, 1969, 1970) and Palmore (1969) give the conjectures. The first results on this problem are in Meyer and Palmore (1970b) where topological methods were used to find new periodic solution in the restricted problem. The complete answer came in Henrard (1970b, 1973) and Schmidt (1973). Henrard (1970b) contains only formal results on the normal form. One must read Henrard (1973) to get the analytic proof. Earlier partial results along these lines are found in Alfriend (1970, 1971) and Roels (1971a, 1971b); related works are Ito (1984), Sanders (1977), Sanders and Verhulst (1979), Schmidt and Sweet (1973), and Sweet (1973); and later work which rediscovers these results are Contopoulos (1981) and Duistermaat (1984).

Recently Chow (1985) has begun the study of the bifurcations of periodic solutions of a Hamiltonian system which depends on one parameter. There are many cases and some of these cases are not completely analyzed. However, Chow's study along with the works of Henrard and Schmidt discussed above carry us a long way toward a complete picture of codimension one bifurcations in Hamiltonian systems.

When there is more than one parameter the number of cases increase dramatically. Some partial results are found in Stellmacker (1984a, 1984b) and van der Aa (1983).

There are few books that treat bifurcations of periodic solutions of Hamiltonian systems in any detail Abraham and Marsden (1978) gave a lively discussion but no proofs. Sanders and Verhulst (1985) give careful proofs but don't cover much material. Markeev (1978) and Bruno (1979) discuss periodic solutions of Hamiltonian systems in some detail, but do not cover all the bifurcations of interest here.

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