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AN ANOSOV TYPE STABILITY THEOREM FOR ALMOST PERIODIC SYSTEMS

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ABSTRACT. In this paper I discuss a natural generalization of the structural stability theorem for Anosov diffeomorphisms i.e. diffeomorphisms which have a global hyperbolic structure. The maps discussed define skew product dynamical systems over a discrete almost periodic system. This is the natural generalization for almost periodic systems of the Poincaré map for periodic systems. This follows from the Miller-Sell method of embedding an almost periodic system of differential equations in a flow. Generalizations are given of the shadowing lemma, the expansive property, and the openness and the structural stability of Anosov systems.

I. Introduction. Recently, George Sell and I have been developing a geometric theory of systems of almost periodic (a.p.) differential equations along the lines suggested by Smale (1967) for autonomous or periodic systems. Smale's program seeks global stability results and rest heavily on the concept of a hyperbolic structure. One of the main tools of this theory is the shadowing lemma of Anosov (1967) and Bowen (1975).

Miller (1965) and Sell (1967) showed how to embed the solutions of an almost periodic system of differential equations in a dynamical system . This dynamical system is a skew product flow over the translation flow on the hull of the a.p. equations. This embedding introduces geometric techniques into the theory of a.p. systems.

In Meyer and Sell (1987a), we present a simple analytic proof of the classical shadowing lemma which easily generalizes to the skew product systems of Miller and Sell. In Meyer and Sell (1987c) we present a slightly different generalization of the shadowing lemma. In Meyer and Sell (1987b,c), we give a generalization the Smale horseshoe basic set and Melnikov's method to a.p. systems. This paper will give a generalization to a.p. systems of the Anosov

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(1967) stability theorem. The proof of this stability theorem is a simple application of the generalized shadowing lemma given in our previous papers once we establish that the generalized Anosov systems are open in the appropriate topology.

Although one usually thinks of Smale's program as dealing solely with dissipative systems, both the horseshoe and Anosov systems appear in Hamiltonian systems. In fact Poincaré (1899) discussed transverse homoclinic orbits, which imply horseshoes, in the restricted three body problem. Geodesic flows on manifolds with negative curvature are Hamiltonian Anosov systems -- see Anosov (1967). Markus and Meyer (1974) give another example of Hamiltonian Anosov system.

The Section II gives a brief introduction to some of the basic geometric results about almost periodic systems. In particular the hull of an a.p. function, the translation flow on the hull, the existence of cross sections, and almost periodic suspensions are defined and discussed. Section III gives the Miller-Sell embedding of the solutions of a system of a.p. equations into a skew product dynamical system. It also gives the definitions of a skew Anosov system, skew equivalence and skew structural stability. With these definitions the main theorem says the skew Anosov systems are skew structurally stable. Section IV contains a discussion of the shadowing lemma for skew Anosov systems, the proof of the openness of skew Anosov systems and the proof of the structural stability of skew Anosov systems using these two facts.

II. The Hull, Cross Sections, and Suspensions. Throughout this paper *almost periodic* (a.p.) will be in the sense of Bohr(1959). Besicovitch (1932), Bohr (1959), Favard (1933) and Fink (1974) are good general references on almost periodic functions and differential equations. The examples and some of the other elementary facts given here are discussed in more detail in Meyer and Sell (1987c). Let $\mathbf{C} = \mathbf{C}(R, R^n)$ (or $\mathbf{C}(R, C^n)$) denote the space of continuous functions from R into R^n (or C^n) with the topology of uniform convergence on compact set -- the compact open topology. Translations define a flow on $\mathbf{C}$ as follows

$$(1) \qquad \pi : \mathbf{C} \times R \longrightarrow \mathbf{C} : (f, \tau) \rightarrow f_\tau$$

where $f_\tau(t) = f(t+\tau)$. For any $f \in \mathbf{C}$ the orbit closure of f is called the *hull* of f and is denoted by H(f). If f is a.p. then H(f) is a compact minimal set; each element $g \in H(f)$ is a.p. with $H(f) = H(g)$; $\pi|H(f)$ is equicontinuous; and

H(f) can be given a compact, connected Abelian group structure. We also let **AP** denote the space of a.p. functions with the sup norm on R. The hull is defined in this space and the above results hold there also -- see Sell (1971).

If f is a.p., its associated Fourier series will be denoted by

(2) $$f \sim \sum a_k \exp i\omega_k t.$$

It follows that $f_\tau \sim \sum a_k \exp i\omega_k(t+\tau)$. If $f_{\tau_n} \to g$, use the Cantor diagonal procedure to select a subsequence if necessary such that

(3) $$\tau_n \to \alpha_k \mod 2\pi/\omega_k \text{ as } n \to \infty, \text{ for all } k.$$

Then the Fourier coefficients of f_τ converge to the Fourier coefficients of g so

(4) $$g \sim \sum a_k \exp i\omega_k(t+\alpha_k).$$

Thus, if $g \in H(f)$ there are angles α_k defined mod $2\pi/\omega_k$ such that (4) holds.

Example 1: Consider a quasi-periodic function of the form

(5) $$q(t) = a_1 \exp i\omega_1 t + a_2 \exp i\omega_2 t$$

where ω_1/ω_2 is irrational and a_1, a_2 are real. In this case

$$H(q) = \{\, a_1 \exp i\omega_1(t+\alpha_1) + a_2 \exp i\omega_2(t+\alpha_2) : \alpha_i \text{ defined mod } 2\pi/\omega_i \,\}.$$

Thus the two angles α_1, α_2 are coordinates for H(q), or H(q) is homeomorphic to the two torus.

Example 2: Consider a limit periodic function of the form

(6) $$\ell(t) = \sum_0^\infty a_k \exp i2\pi \left[\frac{t}{2^k} \right]$$

where the a_k are chosen so that the series converges absolutely and uniformly. In this case $g \in H(\ell)$ if and only if

(7) $$g(t) = \sum_0^\infty a_k \exp i2\pi \left[\frac{t + \alpha_k}{2^k} \right]$$

where the angles α_k are defined mod 2^k and satisfy $\alpha_k \equiv \alpha_{k+1} \mod 2^k$. In this case $H(\ell)$ is homeomorphic to the standard solenoid -- see Figure 1.

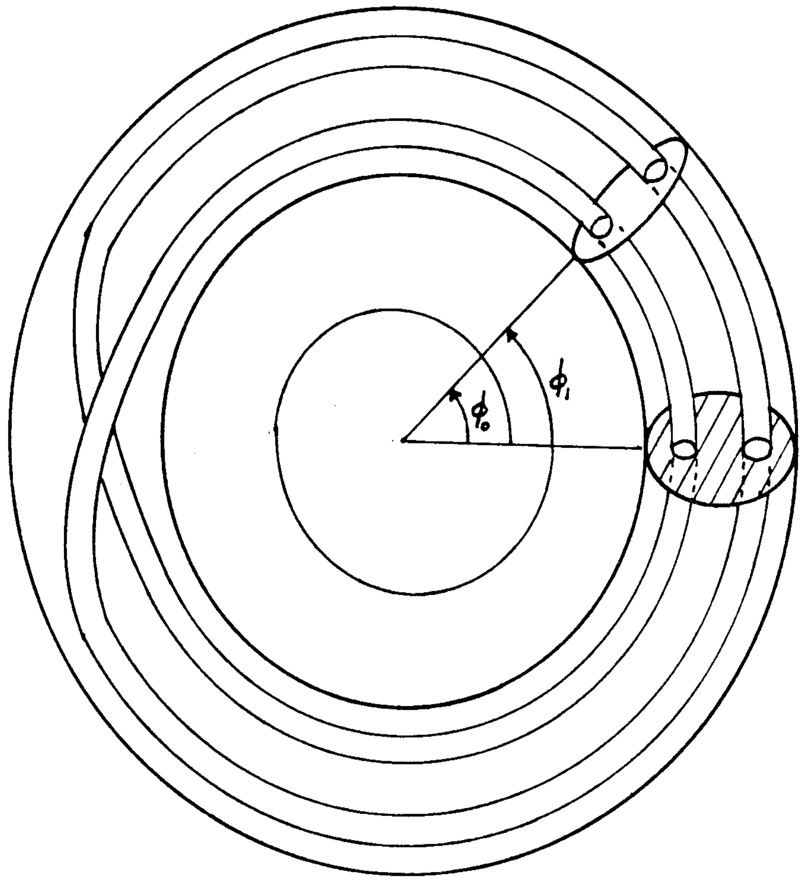

Figure 1. $H(\ell)$ -- The solenoid.

A flow $\sigma : X \times R \to X$, X a compact metric space, admits a *(global) cross section* Z if i) Z is a closed subset of X, ii) all trajectories meet Z, and iii) there is a positive continuous function $T : Z \to R$ such that $\sigma(z,T(z)) \in Z$ for all $z \in Z$ and $\sigma(z,t) \notin Z$ for $0 < t < T(z)$. The function T is called the *first return time*. The *Poincare map* (or *section map*) is the map

(8) $P : Z \longrightarrow Z : z \longrightarrow \sigma(z, T(z))$.

The translation flow on the hull of an almost periodic function always admits a cross section. Let f be a.p. and have a Fourier series given in (2) then if $g \in \mathbf{H}(f)$

(9) $$g_\tau \sim \sum a_k \exp i\omega_k(t+\alpha_k+\tau) \sim \sum a_k \exp i\omega_k(\alpha_k+\tau) \exp i\omega_k t$$

thus the Fourier coefficient corresponding to the frequency ω_k is $a_k \exp i\omega_k(\alpha_k+\tau)$ which is has a constantly changing argument as τ varies provided $\omega_k \neq 0$. Thus a cross section to the translation flow on $\mathbf{H}(f)$ is

(10) $$Z = \{ g \in \mathbf{H}(f) : \arg (a_k \exp i\omega_k(\alpha_k+\tau)) = 0 \}.$$

In this case the first return time is $2\pi/\omega_k$ and the Poincaré map defines a discrete a.p. dynamical system.

Example 1. A cross section to the translation flow on $\mathbf{H}(q)$ is $\alpha_1 \equiv 0 \bmod 2\pi/\omega_1$ and α_2 can be used as a coordinate it this cross section. In this case the Poincaré map is the irrational rotation of the circle $P : \alpha_2 \longrightarrow \alpha_2 + (\omega_2/\omega_1)2\pi$.

Example 2. A cross section to the translation flow on $\mathbf{H}(\ell)$ is $\phi_1 \equiv 0 \bmod 1$ -- the shaded disk in Figure 1. Topologically, this cross section is a Cantor set and the associated Poincaré map is equivalent to the classical adding machine. The adding machine is the dynamical system

(11) $$\gamma : \prod_0^\infty \{0,1\} \longrightarrow \prod_0^\infty \{0,1\} : \ldots a_2a_1a_0. \longrightarrow \ldots a_2a_1a_0. + 1,$$

i.e. the space is all binary integers with the product topology and the map adds 1 to a binary integer. See Meyer and Sell (1987c) for more details.

Let $P : Z \longrightarrow Z$ be a discrete a.p. dynamical system, say the irrational rotation of the circle or the adding machine. Let $D : X \longrightarrow X$ be a discrete dynamical system, i.e. D is a homeomorphic of the topological space X. The *P - almost periodic suspension of* D is defined as the suspension of the product system $P \times D : Z \times X \longrightarrow Z \times X : (z,x) \longrightarrow (P(z),D(x))$. That is, first define the parallel flow

$$\gamma : (Z \times X \times R) \times R \to (Z \times X \times R) : ((z,x,\tau),t) \to (z,x,\tau+t)$$

and then drop this flow to the quotient space $(Z \times X \times R)/\sim$ where $\sim$ is the equivalence relation $(z,x,\tau) \sim (P(z),D(x),\tau+1)$.

III. Skew Product Flows and Skew Anosov Systems. Now let **G** be the space of functions f from $R^n \times R^1$ into R^n such that for every compact set $K \subset R^{2n}$, (i) the function is uniformly continuous on $K \times R$ and (ii) there is a constant k such that

$$|f(x,t) - f(y,t)| < k|x - y|, \quad t \in R, \ x,y \in K.$$

Let **G** be given the compact open topology. Define the flow $\pi : \mathbf{G} \times R \to \mathbf{G}$: $(F,\tau) : \to F_\tau$ where $F_\tau(x,t) = F(x,t+\tau)$ and define the hull as before. Let $L(x,t) \in \mathbf{G}$ be almost periodic in t uniformly in x (u.a.p.). Consider the system of differential equation

$$(1) \qquad \dot{x} = F(x,t), \quad F \in \mathbf{H}(L).$$

This might be a Hamiltonian system on an even dimensional space. Let $\Phi(t,x,F)$ be the solution of (1) such that $\Phi(0,x,F) = x$. Assume that Φ is defined for all $t \in R$, $x \in R^n$, $F \in \mathbf{H}(L)$. Miller (1965) defined a flow on $R^n \times \mathbf{H}(L)$ by

$$(2) \qquad \begin{aligned} \Pi : (R^n \times \mathbf{H}(L)) \times R &\to R^n \times \mathbf{H}(L) \\ : ((x, F), t) &\to (\Phi(t,x,F), F_t). \end{aligned}$$

This is an example of a skew product flow, where the space is a product and the flow acting on the second factor is a flow in its own right. Under the general assumption of smooth F in (1) the function Φ and its first partial with respect to x will be continuous on $R^n \times \mathbf{H}(L)$, but it makes no sense to speak of a partial derivative of Φ with respect to F because $\mathbf{H}(g)$ is not a manifold in general. See Sell (1971) for a general discussion and more details.

Example 3. Consider the differential equation

$$(3) \qquad \dot{x} = f(x) + \varepsilon p(t), \qquad p \in \mathbf{H}(r)$$

where $f : R^n \to R^n$ is smooth and $r : R \to R^n$ is a.p. Also assume that solutions are defined for all t and for all values of the small parameter ε. Let Z be any cross section for the flow on the hull of r with constant first return time T and Poincaré map P. Let $\phi(t,x)$ be the solution of (3) such that

$\phi(0,x) = x$ when $\varepsilon = 0$. In order to do a perturbation analysis one considers (3) as an a.p. system even when $\varepsilon = 0$. That being the case, when $\varepsilon = 0$ the dynamical system defined by (2) equivalent to the P-almost periodic suspension of $\phi(T,x)$. Thus we can consider (3) as a perturbation problem where the unperturbed system is a P-almost periodic suspension. Notice that in this example the perturbation would not change the flow on the base, i.e. the translation flow on the hull of r would be the same for all values of the perturbation parameter ε. This is the motivation for the definitions given below.

Let $P : Z \to Z$ be a discrete a.p. dynamical system and M a smooth, connected, compact manifold. Then $A : M \times Z \to M \times Z$ will be called a *skew Anosov system (over P)* if

i) A is a skew product system over P, i.e. $A(m,z) = (B(m,z),P(z))$;

ii) $B : M \times Z \to M$, has a continuous partial derivative with respect to its first argument, denoted by D_1B;

iii) there exist subspaces $E^s_{(m,z)}$ and $E^u_{(m,z)}$ such that

$$T_mM = E^s_{(m,z)} \oplus E^u_{(m,z)} \text{ for all } (m,z) \in M \times Z \tag{4}$$

and this splitting is continuous;

iv) $D_1B(m,z) : E^s_{(m,z)} \to E^s_{A(m,z)} \quad D_1B(m,z) : E^u_{(m,z)} \to E^u_{A(m,z)}$

v) there are constants $C > 0$ and $0 < \lambda < 1$ such that

$$\begin{aligned} &\| D_1B^n(m,z)(u) \| \le C\,\lambda^n \| u \| \text{ for } u \in E^s_{(m,z)} \text{ and } n > 0 \\ &\| D_1B^{-n}(m)(u) \| \le C\,\lambda^n \| u \| \text{ for } u \in E^u_{(m,z)} \text{ and } n > 0, \end{aligned} \tag{5}$$

and all $(m,z) \in M \times Z$.

Let $A_i : M \times Z \to M \times Z : (m,z) \to (B_i(m,z),P(z))$ $i = 1,2$ be two skew product systems over the same base $P : Z \to Z$. We say A_1 and A_2 are *skew equivalent* if there is a homeomorprism $H : M \times Z \to M \times Z$ $: (m,z) \to (h(m,z),z)$ such that the following diagram commutes:

$$(6)\qquad \begin{array}{ccc} M \times Z & \xrightarrow{A_1} & M \times Z \\ \downarrow & & \downarrow H \\ M \times Z & \xrightarrow[A_2]{} & M \times Z \end{array} .$$

Note that here and several times below we treat the second variable differently. Here we require that H be the identity map on the second factor. Thinking about the differential equation examples given above these seems natural since the second factor corresponds to the time translate of the equations. Thus H does not change the clock.

Let $C_P^1 = C_P^1(M \times Z,\ M \times Z)$ be the space of functions $\Phi : M \times Z \to M \times Z : (m,z) \to (\phi(m,z),P(z))$ where ϕ has a continuous first partial with respect to it first argument and we place the topology of uniform convergence of the functions and there first partial with respect to its first argument. That is two such functions are close if their values are close and their first partials are close. We say that $\Phi \in C_P^1$ is *skew structurally stable* if the is a neighborhood **N** of Φ in C_P^1 such that if $\Psi \in$ **N** then Φ and Ψ are skew equivalent. The main result of this note is:

Theorem: Skew Anosov systems are skew structurally stable.

IV. The Shadowing Lemma, Openness, and the Proof of Structural Stability. Let $P : Z \to Z$ be a discrete a.p. dynamical system, M a smooth compact, connected manifold and $A : M \times Z \to M \times Z : (m,z) \to (B(m,z),P(z))$ be a discrete skew product dynamical system. For $\alpha > 0$ a *(skew) α-pseudo-orbit* for A is a bisequence $\{(m_i,z_i)\}$, $-\infty < i < \infty$, with $z_{i+1} = P(z_i)$ and $d(m_{i+1},B(m_i,z_i)) < \alpha$ for all i. Here d is some distance function on M. Note that $\{z_i\}$ is a P-orbit and so we allow jumps of distance α in the M direction only. If we think in terms of the differential equation examples of the previous section this means we allow jumps in the solutions of one equation but do not allow a jump in the equations. An A-orbit $\{ A^i(m_0,z_0) = (m_i,z_i) \}$ *(skew) β-shadows* an α-pseudo-orbit $\{ (p_i,z_i) \}$ if $d(m_i,p_i) < \beta$ for all i and of course $z_{i+1} = P(z_i)$. Note that the base orbits are the same. In Meyer and Sell (1987c) we give a simple proof of:

Theorem (The skew shadowing lemma): If A is an skew Anosov system, then for every $\beta > 0$ there is an $\alpha > 0$ such that every α-pseudo-orbit is β-shadowed by an A-orbit. Moreover, there is a $\beta_0 > 0$ such that if $0 < \beta < \beta_0$ then the A-orbit given above is uniquely and continuously determined by the α-pseudo-orbit.

Continuity means that the map which sends $p_0 \longrightarrow m_0$ is continuous. The constant β_0 is a function of the constants C and λ in the definition of an Anosov system.

A is *(skew) expansive* if there is an $\varepsilon > 0$ such that given any two A-orbits $\{ A^i(m,z) \}$ and $\{ A^i(p,z) \}$ with $m \neq p$ there is some j such that $d(B^j(m,z), B^j(p,z)) > \varepsilon$. Note that the second argument is the same. Again thinking in terms of the differential equations the expansiveness is for the solutions of one equation. In Meyer and Sell (1987c) an immediate corollary of the proof of the skew shadowing lemma is:

Corollary: Skew Anosov systems are skew expansive.

In fact the ε can be taken as the β_0 of the shadowing lemma and therefore is a function of the constants C and λ in the definition of an Anosov system.

Here we shall give a new definition of skew Anosov which is different from the one given in the previous section. In the old definition the manifold M was given one Riemannian metric and the estimates in III.6 contained a constant C. In the new definition we assume that $A : M \times Z \longrightarrow M \times Z : (m,z) \longrightarrow (B(m,z), P(z))$ satisfies conditions i), ii), iii), and iv) of the old definition but change v). Now assume that for each $z \in Z$, M is given a metric $(\, , \,)_z : TM \times TM \longrightarrow R$ which varies continuously with z and which in tern defines a norm $\| \ \|_z : TM \longrightarrow R$. Assume there is a constant $0 < \lambda < 1$ such that

v') $\| D_1B(m,z)(u) \|_{P(z)} < \lambda \| u \|_z$ for $u \in E^s_{(m,z)}$

$\| D_1B^{-1}(m,z)(u) \|_{P^{-1}(z)} < \lambda \| u \|_z$ for $u \in E^u_{(m,z)}$,

and all $(m,z) \in M \times Z$.

Lemma: The new and old definition of skew Anosov system are equivalent.

Proof. That the old definition implies the new is proved precisely in the same way as Proposition 4.2 of Shub (1987). Assume that A satisfies the new definition as given above and fix $w \in Z$. Since M and Z are compact and the metric varies continuously there is a constant $K \geq 1$ such that

(1) $$K^{-1} \| u \|_z < \| u \|_w < K \| u \|_z,$$

for all $u \in T_pM$, $p \in M$ and $z \in Z$. Iterating (1) for $u \in E^s_{(m,z)}$ gives

$$\| D_1B^n(m,z)(u) \|_{P^n(z)} < \lambda^n \| u \|_z$$

for $u \in E^s_{(m,z)}$ and using (2) gives

(2) $$K^{-1} \| D_1B^n(m,z)(u) \|_w < K \lambda^n \| u \|_w .$$

And similarly for $u \in E^u_{(m,z)}$. Thus the old definition holds with the single metric $(\ ,\ )_w$ on M with the constant $C = K^2$.

Theorem: The set of Anosov systems is an open set in $C^1_P(MxZ, MxZ)$.

Proof: Let $A : M \times Z \to M \times Z : (m,z) \to (B(m,z),P(z))$ be an Anosov diffeomorphism by the new definition given above and $A' : M \times Z \to M \times Z : (m,z) \to (B'(m,z),P(z))$ be close to A in the C^1_P topology. Let $\mathcal{F}^1 = \mathcal{F}^1(M,Z)$ be the space of C^1 vector field depending on a parameter $z \in Z$, i.e. $X \in \mathcal{F}^1$ if $X : M \times Z \to TM \times Z : (m,z) \to (Y(m,z),z)$ is continuous, has a continuous partial derivative with respect to it first argument, denoted by D_1X, and $Y(m,z) \in T_mM$ for all $(m,z) \in M \times Z$. Place on $\mathcal{F}^1$ the topology of uniform convergence of functions and their first partial derivative with respect to their first argument. Define mappings F, F' : $\mathcal{F}^1 \to \mathcal{F}^1$ by the formulas:

(3)
$$F(X)(m,z) = (D_1B(A^{-1}(m,z))(Y(A^{-1}(m,z)),z) = (G(X)(m,z),z)$$
$$F'(X)(m,z) = (D_1B'(A'^{-1}(m,z))(Y(A'^{-1}(m,z)),z) = (G'(X)(m,z),z)$$

The tangent bundle $TM \times Z = \cup\, T_mM \times Z$ (union on $m \in M$) admits a decomposition

$$(4)\qquad \begin{aligned} &TM \times Z = E^s \oplus E^u \\ &E^s = \cup\, E^s_{(m,z)}, \quad E^u = \cup\, E^u_{(m,z)} \end{aligned}$$

where the latter unions are over all $(m,z) \in M \times Z$. The first factor of F and F', G and G', are linear and so using the splitting (4) we can write

$$(5)\qquad G = \begin{bmatrix} G_{++} & 0 \\ 0 & G_{--} \end{bmatrix}, \qquad G' = \begin{bmatrix} G'_{++} & G'_{+-} \\ G'_{-+} & G'_{--} \end{bmatrix}$$

The matrix for G is diagonal since the splitting is invariant for A. By v') and the fact that we have taken A' close to A in the C^1_p topology it follows that

$$(6)\qquad \begin{aligned} &\| G_{++} u \| < \lambda \| u \| \text{ and } \| G'_{++} u \| < \lambda \| u \| \text{ for } u \in E^s \\ &\| G_{--}^{-1} v \| < \lambda \| v \| \text{ and } \| G'^{-1}_{--} v \| < \lambda \| v \| \text{ for } v \in E^u \\ &\| G'_{+-} v \| < \varepsilon \| v \| \text{ for } v \in E^u, \ \| G'_{-+} u \| < \varepsilon \| u \| \text{ for } u \in E^s \end{aligned}$$

where $0 < \lambda < 1$ and ε can be taken arbitrarily small by taking A' close to A.

Let $\mathcal{L} = \mathcal{L}(E^s, E^u)$ be the space of continuous vector bundle maps with the sup norm, i.e. $L \in \mathcal{L}$, $L(m,z) : E^s_{(m,z)} \rightarrow E^u_{(m,z)}$ is linear. We want to find L so that $\{ (u, Lu) : u \in E^s \}$ is F' invariant subspace. Since

$$(7)\qquad G' \begin{bmatrix} u \\ Lu \end{bmatrix} = \begin{bmatrix} G'_{++} & G'_{+-} \\ G'_{-+} & G'_{--} \end{bmatrix} \begin{bmatrix} u \\ Lu \end{bmatrix} = \begin{bmatrix} G'_{++}u + G'_{+-}Lu \\ G'_{-+}u + G'_{--}Lu \end{bmatrix}$$

invariance takes the form

$$(8)\qquad L\, G'_{++} + L\, G'_{+-} L = G'_{-+} + G'_{--} L$$

or

$$(9)\qquad L = G'^{-1}_{--} \{ -G'_{-+} + L\, G'_{++} + L G'_{+-} L \}$$

Define an operator $\mathcal{T} : \mathcal{L} \to \mathcal{L}$ by

$$(10) \qquad \mathcal{T}(L) = G'^{-1}_{--} \{ -G'_{-+} + L\, G'_{++} + LG'_{+-}L \}$$

so a fixed point of $\mathcal{T}$ solves (10). Let

$$(11) \qquad \mathbb{L} = \{ L \in \mathcal{L} : \| L \| = \sup_{(m,z)} \sup_{\|x\|=1} \| L(m,z)(x) \| \le 1 \}.$$

If $L \in \mathbb{L}$ then

$$(12) \qquad \begin{aligned} \| \mathcal{T}(L) \| &\le \| G'^{-1}_{--} \| (\| G'_{-+} \| + \| L \| \| G'_{++} \| + \| L \|^2 \| G'_{+-} \| \\ &\le \lambda (\varepsilon + \lambda + \varepsilon) \le 1 \end{aligned}$$

provided ε is sufficiently small, so $\mathcal{T} : \mathbb{L} \to \mathbb{L}$. Furthermore, for $L, K \in \mathbb{L}$

$$(13) \qquad \begin{aligned} \| \mathcal{T}(L) - \mathcal{T}(K) \| &\le \| G'^{-1}_{--} \| \{ \| L - K \| \| G'_{++} \| + \| LG'_{+-}L - KG'_{+-}K \| \} \\ &\le \lambda \{ \lambda \| L - K \| + \| LG'_{+-}(L-K) \| + \| (K-L)G'_{+-}K \| \\ &\le \lambda \{ \lambda + 2\varepsilon \} \| L - K \| \end{aligned}$$

and so for ε sufficiently small $\mathcal{T}$ is a contracting map which has a unique fixed point L in $\mathbb{L}$.

Thus we have constructed a bundle $E'^s = \{ (u, Lu) : u \in E^s \}$ which is F' invariant. The bundle $E'^u = \{ (Ku, u) : u \in E^u \}$ is constructed in a similar manner. By construction both K and L have norm less than 1 and the dimensions of the fibers of E'^s and E^s are the same as are those of E'^u and E^u. If $v = (v^s, v^u) \in E'^s_{(m,z)} \cap E'^u_{(m,z)}$ then $v^u = Lv^s = LKv^u$ but since the norms of L and K are less that 1 this implies $v^u = v^s = 0$. Thus $TM \times Z = E'^s \oplus E'^u$. The estimates of the form (1) follow at once from the inequalities (7).

Proof of the structural stability of Anosov systems.

Let A be an Anosov system where $A : M \times Z \to M \times Z : (m,z) \to (B(m,z), P(z))$ and first fix α so that all functions in this α-neighborhood of A are Anosov with the same constants C and λ. Let $\varepsilon > 0$ be the uniform expansive constant and β_0 the uniform constant of the shadowing lemma for all functions in this neighborhood. Let $\beta = \min(\varepsilon/3, \beta_0/3)$ and restrict α further if necessary so that the conclusion of the shadowing lemma holds for this α and β and $\alpha < \beta_0$. Let $D : M \times Z \to M \times Z : (m,z) \to (E(m,z),P(z))$ be within this α of neighborhood of A. Let $(m,z) \in M \times Z$ be arbitrary.

Then since A and D are α close $\{ D^i(m,z) \}$ is an α-pseudo-orbit for A and so there exists a $y = h(m,z)$ such that the A-orbit $\{ A^i(y,z) \}$ β-shadows $\{ D^i(m,z) \}$. The function $h : M \times Z \to M$ is continuous by the shadowing lemma and hence so is $H : M \times Z \to M \times Z : (m,z) \to (h(m,z),z)$. Let $(m,z) \neq (m',z')$. Clearly if $z \neq z'$ $H(m,z) \neq H(m',z')$ so let $z = z'$ and $m \neq m'$. By the expansive property of D there is a j such that $d(E^j(m,z),E^j(m',z)) > \varepsilon$. But $d(E^j(m,z),B^j(y,z)) < \beta \le \varepsilon/3$ and $(E^j(m',z),B^j(y',z)) < \beta \le \varepsilon/3$ and so $d(B^j(y,z),B^j(y',z)) > \varepsilon/3$ or $y \neq y'$. Therefore h and H are one to one. Thus for fixed $z \in Z$ the map $h(.,z) : M \to M$ is a continuous, one-to-one mapping of a compact, connected Hausdorff space and so is a homeomorphism. This implies that H is a homeomorphism also.

Since $d(E^i(m,z),B^i(y,z)) < \alpha$ for all i we have

$$d(E^{i-1}(E(m,z),z),B^{i-1}(B(y,z),z) = d(E^i(m,z),B^i(y,z)) < \alpha < \beta_0.$$

Thus the A orbit through $A(y,z) = (B(y,z),z)$ β_0-shadows the D-orbit through $D(m.z) = (E(m,z),z)$and so by uniqueness $A(y,z) = H(D(m,z))$. But $(y,z) = H(m,z)$ so $A \circ H = H \circ D$ or H is a skew equivalence.

References

Anosov, D. V. 1967 : *Geodesic flows on compact Riemannian manifolds of negative curvature*, **Proc. Steklov Math. Inst.** 90.

Besicovitch, A.S. 1932: **Almost Periodic Functions,** Cambridge Univ. Press.

Bohr, H. 1959: **Almost Periodic Functions**, Chelsea Publ., New York.

Bowen, R. 1975: *ω - limit sets for Axiom A diffeomorphisms*, **J. Diff. Eqs.** 18, 333-9.

Favar, J. 1933: **Lecons sur les fonctions presque periodiques**, Gauthier-Villars, Paris.

Fink, A.M. 1975: **Almost Periodic Differential Equations**, Lect. Notes in Math. 377, Springer-Verlog, New York.

Markus, L. and Meyer, K.

1974: *Generic Hamiltonian systems are neither integrable nor dense*, **Mem. Amer. Math. Soc.** 144.

1980: *Periodic orbits and solenoids in generic Hamiltonian systems*, **Amer. J. Math.** 102, 25-92.

Mather, J. 1968: *Characterization of Anosov diffeomorphisms*, **Nederl. Akad. Wetensch. Indag. Math.** 30, 479-95.

Meyer, K. R. and Sell, G. R.

1987a: *An analytic proof of the shadowing lemma*, to appear in **Funkcialaj Ekvacioj.**

1987b: *Homoclinic orbits and Bernoulli bundles in almost periodic systems*, **Oscillations, Bifurcations and Chaos**, CMS-AMS 8, Providence, R.I., 527-44.

1987c: *Melnikov transforms, Bernoulli bundles, and skew product flows*, to appear.

Miller, R. K. 1965: *Almost periodic differential equations as dynamical systems with applications to the existence of a.p. solutions*, **J. Diff. Eqs.** 55, 337-45.

Poincaré, H. 1899: **Les methods nouvelles de la mechanique celeste III**, Gauthier-Villar, Paris.

Poinkoff, H. G. 1973: *An analytic closing lemma*, **Proc. Midwest Dyn. Sys. Sem.**, Northwestern Univ. Press, Evanston, Ill, 128-256.

Sell, G. R.

1967: *Nonautonomous differential equations and topological dynamics I, II*, **Trans. Amer. Math. Soc.** 127, 241-62 & 263-83.

1971: **Topological Dynamics and Ordinary Differential Equations**, Van Nostrand, New York.

Shub, M. 1987: **Global Stability of Dynamical Systems**, Springer-Verlog, New York.

Smale, S. 1967: *Differentiable dynamical systems*, **Bull. Amer. Math. Soc.** 73, 747-817.

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