The Geometry of Harmonic Oscillators†

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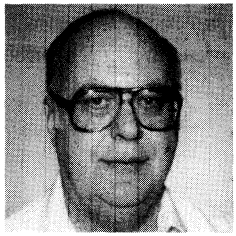

KENNETH R. MEYER: I received my Ph.D. from the University of Cincinnati in 1964, was an Assistant Professor at Brown University, was an Associate Professor at the University of Minnesota and am Professor at the University of Cincinnati. I learned the geometry of harmonic oscillators from the late Charlie Conley.

I. Introduction. About a year ago an engineering colleague dropped by my office to ask some questions about the analysis of a system of differential equations. After hearing the problem, I said he needed to know about the geometry of pairs of harmonic oscillators. He looked a little hurt at my suggestion, but consented to indulge me for a while. After I lectured at the chalk board for about an hour, he admitted that before my lecture he had thought he understood the harmonic oscillator, but now it was clear that he hadn't.

Although an introduction to the harmonic oscillator comes early in the study of the calculus and it is extensively used throughout physics, the beautiful geometry of harmonic oscillators is seldom given in differential equations texts of any level. Even some researchers in oscillation theory are unaware of the interrelationship between the Hopf fibration, the ergodic flow on a torus, the rational approximations of irrational numbers, the topology of the 3-sphere and the lowly harmonic oscillator. The material presented below has been known since the time of Poincaré and Birkhoff and so nothing given here is new. It is essential in Conley's analysis of periodic solutions of the restricted three body problem [**4**], Weinstein's normal mode theorem [**11**] and the generalizations of these theorems found in [**5**, **7**, **8**].

First I will present the geometry of a single harmonic oscillator from a dynamical systems point of view with particular emphasis on the use and interpretation of polar coordinates. Then a model for the 3-sphere as an identification space is obtained by using polar coordinates for a pair of harmonic oscillators. Many basic geometric facts can be gleaned from this simple model of the 3-sphere. The Hopf fibration, the knotting and linking of solutions, and the irrational flow on the torus are a few of the topics presented. The classical regularization of the Kepler problem (the central force problem) is given as an example of a non-trivial physical problem which can be reduced to a pair of harmonic oscillators. Thus the geometry discussed here sheds some light on the classical two body problem.

II. One Harmonic Oscillator. From the point of view of dynamical systems theory, a differential equation defines a flow on a space and the solutions are curves in that space. Historically, the problems come from classical dynamics and so the independent variable is 'time' and will be denoted by t. Newton's notation for fluxions persists with

$$\dot{\ } = \frac{d}{dt} \quad \text{and} \quad \ddot{\ } = \frac{d^2}{dt^2}. \qquad (1)$$

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Thus, the harmonic oscillator,

$$\ddot{x} + \omega^2 x = 0, \tag{2}$$

where ω is a positive constant, is written as a system of first order equations by introducing the new variable $u = \dot{x}/\omega$, to become

$$\dot{x} = \omega u$$
$$\dot{u} = -\omega x. \tag{3}$$

The variable u is a scaled velocity and so the x, u plane is essentially the position-velocity plane, or the phase space of physics. The basic existence and uniqueness theorem of differential equations asserts that through each point (x_0, u_0) in the plane there is a unique solution which passes through this point at any particular epoch t_0. These solutions are given by the formula

$$\begin{pmatrix} x(t, t_0, x_0, u_0) \\ u(t, t_0, x_0, u_0) \end{pmatrix} = \begin{pmatrix} \cos \omega(t - t_0) & \sin \omega(t - t_0) \\ -\sin \omega(t - t_0) & \cos \omega(t - t_0) \end{pmatrix} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix}. \tag{4}$$

The solution curves are parameterized circles. The reason that one introduces the scaled velocity instead of using the velocity itself, as is usually done, is so that the solution curves become circles instead of ellipses. In dynamical systems the geometry of this family of curves in the plane is of prime importance.

The system admits an integral (energy in physical problems)

$$I = x^2 + u^2 \tag{5}$$

which is constant along the solutions of (3), since $\dot{I} = 2x\dot{x} + 2u\dot{u} = 0$ by virtue of (3). Since a solution lies in the set where $I =$ constant, a circle in the x, u plane, the integral alone gives the geometry of the solution curves in the plane. See FIGURE 1.

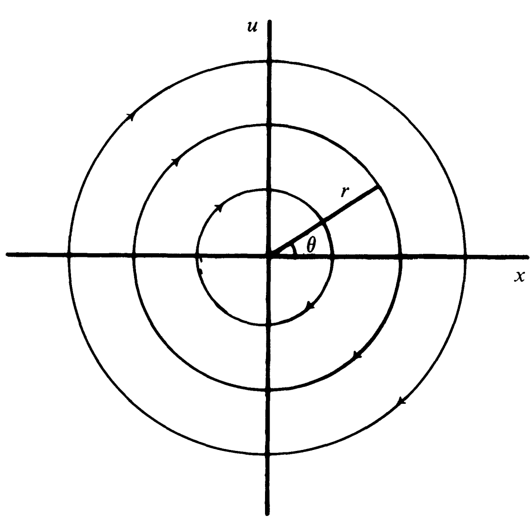


FIG. 1

If we introduce polar coordinates $r^2 = x^2 + u^2$, $\theta = \tan^{-1} u/x$ then the equations (3) become

$$\dot{r} = 0, \qquad \dot{\theta} = -\omega. \tag{6}$$

Here again we see that the solutions lie on circles about the origin since, $\dot{r} = 0$, and that the circles are swept out with constant angular velocity.

In many cases the harmonic oscillator is only a first approximation to a physical problem. For example one gets a harmonic oscillator from the pendulum equation by making the small angle approximation $\sin\theta \simeq \theta$. Some physical problems are better approximated by nonlinear oscillators such as

$$\dot{r} = 0, \qquad \dot{\theta} = -\omega + \alpha r + \cdots, \tag{7}$$

where $\alpha \neq 0$ is a constant. In such cases the solution curves will still be circles, since $\dot{r} = 0$, but the angular frequency, $-\omega + \alpha r + \cdots$, varies from circle to circle.

III. Pairs of Oscillators. Consider a pair of harmonic oscillators

$$\ddot{x} + \omega^2 x = 0, \qquad \ddot{y} + \mu^2 y = 0 \tag{8}$$

which as a system becomes

$$\begin{aligned} \dot{x} &= \omega u & \dot{y} &= \mu v \\ \dot{u} &= -\omega x & \dot{v} &= -\mu y. \end{aligned} \tag{9}$$

In polar coordinates it becomes

$$\begin{aligned} \dot{r} &= 0 & \dot{\rho} &= 0 \\ \dot{\theta} &= -\omega & \dot{\phi} &= \mathbf{x} - \mu \end{aligned} \tag{10}$$

and it admits integrals

$$I = x^2 + u^2 \qquad J = y^2 + v^2. \tag{11}$$

In many physical problems these equations are only the first approximation and the full system does not admit the two individual functions I and J as integrals but does admit the sum

$$E = I + J = x^2 + u^2 + y^2 + v^2 \tag{12}$$

or total energy as an integral. In this case the 3-sphere, $S^3 = E^{-1}(1)$, is an invariant set for the flow, (i.e. a solution which starts in this set stays in this set.) Think for example of a pea, of unit mass, rolling around in a bowl above the x-y plane with its minimum at the origin. If the only force acting on the pea is the constant force due to gravity, the potential, $V(x, y)$ of the pea would have a minimum at the origin. Let V have a Taylor expansion of the form $2V(x, y) = x^2 + y^2 + \cdots$. Then the equations of motion of the pea are $\ddot{x} = -V_x = -x + \cdots$, $\ddot{y} = -V_y = -y + \cdots$ and total energy is $(\dot{x}^2 + \dot{y}^2)/2 + (x^2 + y^2 + \cdots)/2$. Thus the linearized system at the minimum would be of the form (9) with $\omega = \mu = 1$.

Consider the general solution through $r_0, \rho_0, \theta_0, \phi_0$ at epoch $t = 0$. The solutions with $r_0 = 0$ and $\rho_0 > 0$ or $\rho_0 = 0$ and $r_0 > 0$ lie on circles and correspond to periodic solutions of period $2\pi/\mu$ and $2\pi/\omega$, respectively. These periodic solutions are special and are usually called the normal modes.

The set where $r = r_0 > 0$ and $\rho = \rho_0 > 0$ is an invariant torus for (9) or (10). Angular coordinates on this torus are θ and ϕ and the equations are

$$\dot{\theta} = -\omega \qquad \dot{\phi} = -\mu, \tag{13}$$

the standard linear equations on a torus.

If θ/μ is rational then $\omega = p\tau$ and $\mu = q\tau$ where p and q are relatively prime integers. In this case the solution of (13) through θ_0, ϕ_0 at epoch $t = 0$ is $\theta(t) = \theta_0 - \omega t$, $\phi(t) = \phi_0 - \mu t$ and so if $T = 2\pi/\tau$ then $\theta(T) = \theta_0 + p2\pi$ and $\phi(T) = \phi_0 + q2\pi$. That is, the solution is periodic with period T on the torus (see FIGURE 2a) and this corresponds to periodic solutions of (9).

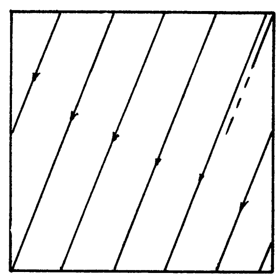
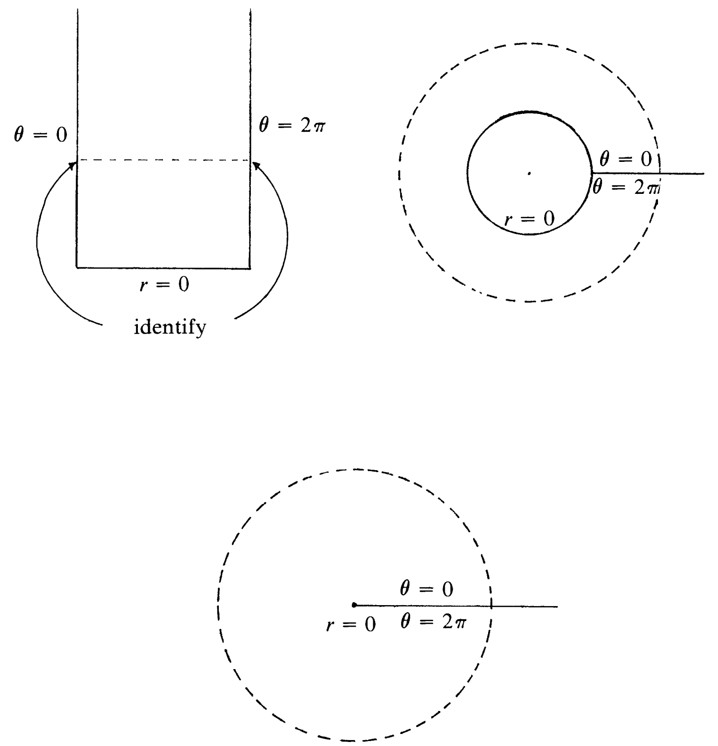


FIG. 2

If θ/μ is irrational then none of the solutions are periodic. In fact the solutions of (13) are dense lines on the torus (see FIGURE 2b) and this corresponds to the fact that the solutions of (9) are quasi-periodic but not periodic. See Coddington and Levinson [**3**] or Hale [**6**] for a discussion of the irrational flow on the torus.

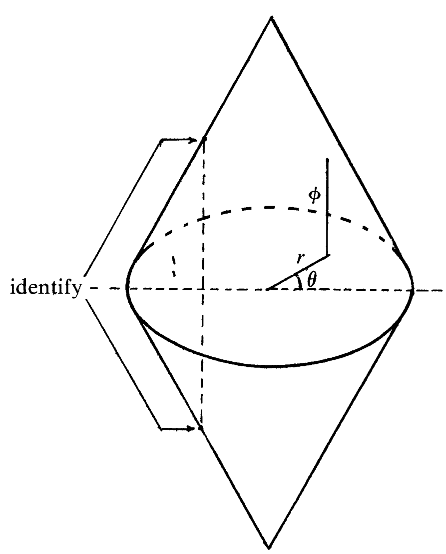
We can use polar coordinates to introduce coordinates on the sphere provided we are careful to observe the conventions of polar coordinates: i) $r \geq 0$, ii) θ is defined modulo 2π and iii) $r = 0$ corresponds to a point. That is, if we start with the rectilinear strip $r \geq 0$, $0 \leq \theta \leq 2\pi$ (FIGURE 3a), identify the $\theta = 0$ and $\theta = 2\pi$ edges to get a half closed annulus (FIGURE 3b), and finally identify the circle $r = 0$ with a point, then we have a plane (FIGURE 3c).

Starting with the polar coordinates, r, θ, ρ, ϕ for R^4 we note that on the 3-sphere $E = r^2 + \rho^2 = 1$ so we can discard ρ and have $0 \leq r \leq 1$. We will use r, θ, ϕ as coordinates on S^3. Now, r, θ with $0 \leqslant r \leqslant 1$ are just polar coordinates for the closed unit disk. For each point of the open disk there is a circle with coordinate ϕ (defined mod 2π), but when $r = 1$, $\rho = 0$ so the circle collapses to a point over the boundary of the disk. The geometric model of S^3 is two solid cones with points on the boundary cones identified as shown in FIGURE 4. Through each point in the open unit disk with coordinates r, θ there is a line segment (the dashed line) perpendicular to the disk. The angular coordinate ϕ is measured on this segment; $\phi = 0$ is the disk, $\phi = \pi$ is the upper boundary cone, $\phi = -\pi$ is the lower boundary cone. Each point on the upper boundary cone with coordinates $r, \theta, \phi = \pi$ is identified with the point on the lower boundary cone with coordinate $r, \theta, \phi = -\pi$. From this model follows a series of interesting geometric facts.



$\theta = 0$
$\theta = 2\pi$
$r = 0$
identify
$\theta = 0$
$\theta = 2\pi$
$r = 0$
$\theta = 0$
$r = 0$
$\theta = 2\pi$

FIG. 3



ϕ
r
θ
identify

FIG. 4

For α, $0 < \alpha < 1$, the set where $r = \alpha$ is a 2-torus in the 3-sphere and for $\alpha = 0$ or 1 the set $r = \alpha$ is a circle. Since r is an integral for the pair of oscillators these tori and circles are invariant sets for the flow defined by the harmonic oscillators. The two circles, $r = 0, 1$ are periodic solutions, called the normal modes. The two circles are linked in S^3, i.e. one of the circles intersects a disk bounded by the other circle in an algebraically non-trivial way. The circle where $r = 1$ is the shaded disk in FIGURE 5 and the circle $r = 0$ intersects this disk once. It turns out that the number of intersections is independent of the bounding disk provided one counts the intersections algebraically. See Rolfsen [**10**].

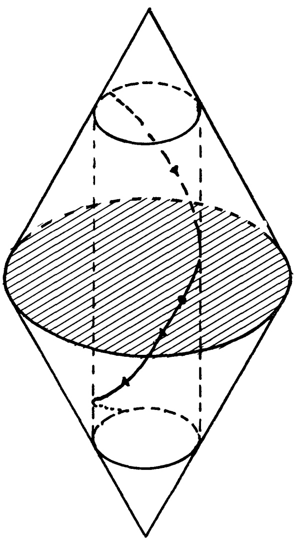


FIG. 5

Consider the special case when $\omega = \mu = 1$. In this case every solution is periodic and so its orbit is a circle in the 3-sphere. Other than the two special circles, on each orbit as θ increases by 2π so does ϕ. Thus each such orbit hits the open disk where $\phi = 0$ (the shade disk in FIGURE 5) in one point. We can identify each such orbit with the unique point where it intersects the disk. One special orbit meets the disk at the center and so we can identify it with the center. The other is the outer boundary circle which is a single orbit. When we identify this circle with a point, the closed disk with its outer circle identified with a point becomes a 2-sphere. Thus:

PROPOSITION. *The* 3-*sphere*, S^3, *is the union of circles. Any two of these circles are linked. The quotient space obtained by identifying each circle with a point, is a* 2-*sphere* (*the Hopf fibration of* S^3).

Let D be the open disk $\phi = 0$, the shaded disk in FIGURE 5. The union of all the orbits which meet D is a product of a circle and a 2-disk, so each point not on the special circle $r = 1$ lies in an open set which is the product of a 2-disk and a circle. By reversing r and ρ in the discussion given above the circle where $r = 1$ has a similar neighborhood. So locally the 3-sphere is the product of a disk and a circle, but the sphere is not the product of a two manifold and a circle. (The sphere has trivial fundamental group but such a product would not. See Rolfsen [**10**].)

When $\omega = p$ and $\mu = q$, p and q relatively prime integers all solutions are periodic and the 3-sphere is again a union of circles but it is not locally a product near the special circles. Arnold [**1**] calls this decomposition into circles near the special circles the Seifert foliation and shows its connection with bifurcation theory. The nonspecial circles are p-q torus knots. They link p times with one special circle and q times with the other.

The linking statements follow by a slight extension of the ideas of the previous proposition. A p-q torus knot is a closed curve which wraps around the standard torus in R^3 in the longitudinal direction p times and in the meridional direction q times. If p and q are different from 1 the knot is nontrivial. We cannot go into details here, but FIGURE 6 shows that the 3-2 torus knot is the classical trefoil or clover-leaf knot. In FIGURE 6a the opposite sides of the squares are to be identified so that it represents a torus. The line of slope $2/3$ in 6a wraps around the torus 3 times in one direction and 2 times in the other. This line is the important object so henceforth the sides of the torus will be suppressed in the drawings. Think of rolling the square in 6a into a cylinder by rolling the top half of 6a back of the bottom half. Now the line wraps over and under the cylinder and has two end points at each end of the cylinder. Join the end points with additional curves to get FIGURE 6b. Note that the crossing patterns for the curves in FIGURES 6b and 6c are the same and so represent the same curve. FIGURE 6c is the standard representation of the trefoil knot. See Rolfsen [**10**] for more information on knots.

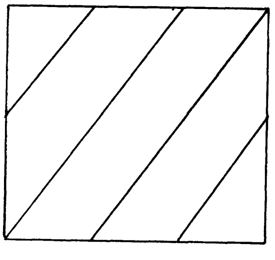


FIG. 6a

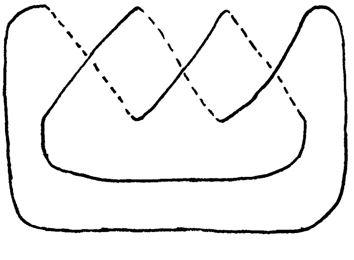


FIG. 6b

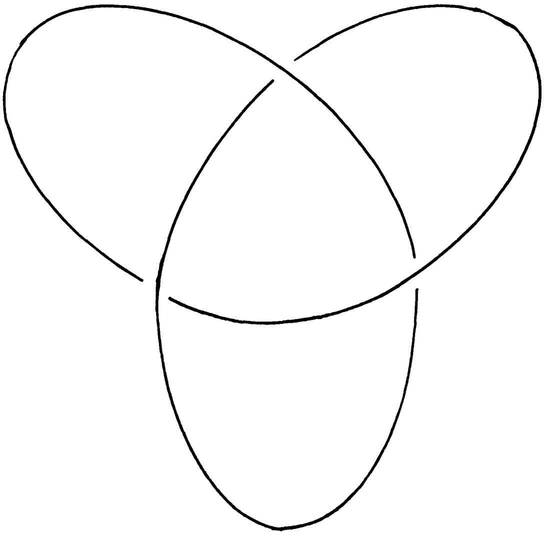


FIG. 6c

Sometimes one needs to consider nonlinear oscillators. A simple and somewhat typical nonlinear system is

$$\begin{aligned} \dot{r} &= 0 & \dot{\rho} &= 0 \\ \dot{\theta} &= \omega + \alpha r & \dot{\phi} &= \mu + \beta\rho, \end{aligned} \tag{14}$$

where α and β are constants. In this case we still have the special circles, the decomposition into tori, $r =$ constant as described above, and the flow on the tori is still linear. However, now the slope of the linear flow on the tori varies from torus to torus. Sometimes all orbits are dense and sometimes all orbits are periodic.

IV. The Kepler Problem. As stated above, pairs of linear oscillators are often the first approximation to interesting nonlinear physical problems, but also they completely solve the Kepler problem of celestial mechanics. Consider a body of mass M fixed at the origin in the plane and another free body of mass m whose position in the plane is given by the vector q. Assume that the free body is attracted to the fixed body by the force given by Newton's law of gravity. Then the equation of motion for the free body is

$$\ddot{q} = \frac{-GMq}{\|q\|^3}, \tag{15}$$

where G is the universal gravitational constant. By changing the time scale if necessary the constant GM can be taken as 1. There are many ways to solve this system of differential equations, but I think one of the most informative ways is the regularization procedure of Levi-Civita. This method removes the singularity at the origin and reduces the equation (15) to a pair of harmonic oscillators with equal frequencies.

Consider q as a complex number, so multiplication makes sense and the norm $\|\cdot\|$ can be replaced with absolute value $|\cdot|$. Since the bounded orbits of the problem occur for negative energy, write the energy integral as

$$|\dot{q}|^2 - 2/|q| = -C, \tag{16}$$

so C is the negative of the total energy. The Levi-Civita change of coordinates is $q = w^2$ together with a change of time $dt = 4|w|^2\,d\tau$. The transformation from w to q is a 2 to 1 transformation except at the origin. After some careful computations the equation of motion in the new coordinates becomes

$$\frac{d^2w}{d\tau^2} + 4Cw = 0, \tag{17}$$

a pair of harmonic oscillators with frequencies $4C$ when $C > 0$. The computations are carried out in full in Szebehely **[9]**.

Thus the flow of the Kepler problem (15) on a negative energy level is the 2 to 1 projection of the flow on the 3-sphere defined by (17) where the projection is the squaring map. In the new coordinates $w = 0$ corresponds to a collision of the two bodies but the equations (17) do not have a singularity there. Thus these coordinates 'regularize the collision singularity'.

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