# Jacobi Elliptic Functions from a Dynamical Systems Point of View

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**1. INTRODUCTION.** The theory of Jacobi elliptic functions arose in an attempt to integrate certain algebraic expressions, but soon found many applications to geometry, mechanics, physics, and engineering. The theory of these important functions is vast, but an interesting introduction with some applications can be given even at the advanced undergraduate level. These functions satisfy a simple system of differential equations, which can be analyzed using some of the basic theory of differential equations. Many of their basic properties come as immediate applications of the fundamental theorems on existence, uniqueness, and continuous dependence of solutions on initial conditions, and thus this system of equations serves as an excellent example in a course in differential equations.

The Jacobi elliptic functions are important examples of doubly periodic meromorphic functions, but the dynamical systems approach that we present considers them as real valued functions only. However, there are many applications of these functions in the real domain.

My first attempts [**10**, pp. 8–10] to give a dynamical systems approach used the second order differential equation presented in Section 2.4, but a far better approach is found in the brief introduction found in Hille's classic book [**9**, pp. 66–74] on the analytic theory of differential equations. It is Hille's definition that I give in Section 2.1. A very nice dynamical systems introduction using this same definition can be found in [**4**, pp. 445–449].

In Section 3, we sketch how to solve a class of differential equations that includes the pendulum equation and the undamped Duffing equation.

Over the years in courses at the undergraduate and graduate level, I have used the Jacobi elliptic functions as an example of the power of the geometric approach to the theory of ordinary differential equations. Rarely do I present all the material in Section 3 and often break up much of the material into a series of problems; but always I present the solution of the pendulum equation.

**2. THE JACOBI ELLIPTIC FUNCTIONS.** For a moment, think of the various definitions of sine and cosine that you have encountered over the years. For many of us, they were first defined as ratios of sides of a right triangle or as coordinates of a point on the unit circle. Some of us were subjected to a rigorous and laborious analytic definition as found in the classic texts by Hardy [**8**, pp. 447–486] or Whittaker and Watson [**11**, pp. 579–590]. A specialist in differential equations might define the sine function as the solution of the harmonic oscillator satisfying $x(0) = 0$, $\dot{x}(0) = 1$ [**1**], or better still define $x(t) = \cos t$, $y(t) = \sin t$ as the solution of a system of first order equations $\dot{x} = y$, $\dot{y} = -x$. It is this latter approach that we take to define the Jacobi elliptic functions.

**2.1. The system definition.** Let $k$ be a number in $(0, 1)$, and let $t$ denote a real variable that we interpret as time. The Jacobi elliptic functions $\mathrm{sn}(t, k)$, $\mathrm{cn}(t, k)$, $\mathrm{dn}(t, k)$

are defined as the solutions of the system of differential equations

$$\begin{aligned} \dot{x} &= yz \\ \dot{y} &= -zx \\ \dot{z} &= -k^2xy \end{aligned} \tag{1}$$

that satisfy the initial conditions

$$\operatorname{sn}(0,k) = x(0) = 0, \qquad \operatorname{cn}(0,k) = y(0) = 1, \qquad \operatorname{dn}(0,k) = z(0) = 1. \tag{2}$$

The dots in (1) denote differentiation with respect to $t$. The parameter $k$ is known as the *modulus* and satisfies $0 < k < 1$; the *complementary modulus* is $\kappa = \sqrt{1-k^2}$. One speaks of these functions by pronouncing the letters as " 's' 'n' of 't' and 'k' ". These functions have also been denoted by $\sin \operatorname{am}(t,k)$, $\cos \operatorname{am}(t,k)$, delta $\operatorname{am}(t,k)$ and called *sine amplitude*, *cosine amplitude* and *delta amplitude*.

The equations (1) are real analytic in the variables $t$, $x$, $y$, $z$ and the parameter $k$, so the basic existence theory of ordinary differential equations ensures that the Jacobi elliptic functions are smooth or even real analytic functions of $t$ and $k$; see [**9**, pp. 48–56, 90–96], [**6**, pp. 18–27], or [**7**, p. 10, 93–113]. The definition immediately gives the derivatives for the functions, namely

$$\begin{aligned} \frac{d}{dt}\operatorname{sn}(t,k) &= \operatorname{cn}(t,k)\operatorname{dn}(t,k), \\ \frac{d}{dt}\operatorname{cn}(t,k) &= -\operatorname{dn}(t,k)\operatorname{sn}(t,k), \\ \frac{d}{dt}\operatorname{dn}(t,k) &= -k^2\operatorname{sn}(t,k)\operatorname{cn}(t,k). \end{aligned} \tag{3}$$

The following theorem is an interesting application of the theorem on the continuous dependence of solutions of a differential equation on parameters.

**Proposition 2.1.** *As $k$ approaches $0$ from the right we have*

$$\operatorname{sn}(t,k) \to \sin(t), \qquad \operatorname{cn}(t,k) \to \cos(t), \qquad \operatorname{dn}(t,k) \to 1, \tag{4}$$

*and as $k$ approaches $1$ from the left we have*

$$\operatorname{sn}(t,k) \to \tanh(t), \qquad \operatorname{cn}(t,k) \to \operatorname{sech}(t), \qquad \operatorname{dn}(t,k) \to \operatorname{sech}(t). \tag{5}$$

*The convergence is uniform on compact sets.*

*Proof.* When $k = 0$, the equations (1) become $\dot{x} = yz$, $\dot{y} = -zx$, $\dot{z} = 0$, and the solutions satisfying $x(0) = 0$, $y(0) = 1$, $z(0) = 1$ are $(\sin(t), \cos(t), 1)$. The solutions of system (1) are continuous in the parameter $k$ for $t$ in a compact set. The limits follow from the theorem on continuous dependence of solutions on parameters; [**9**, pp. 90–96], [**6**, pp. 25–27], or [**7**, p. 94]. This proves (4), and (5) follows in a similar manner. ∎

**2.2. The integrals.** Many of the basic facts about the Jacobi functions are a result of the special properties of equations (1).

**Proposition 2.2.** *The equations (1) admit the two functions*

$$I = x^2 + y^2, \qquad J = k^2x^2 + z^2,$$

*as integrals, i.e., the functions I and J are constant along solutions of (1).*

*Proof.*

$$\frac{d}{dt}I(x(t), y(t)) = 2x\dot{x} + 2y\dot{y} = 2x(yz) + 2y(-zx) \equiv 0,$$

so $I$ is constant along solutions. In the same way, $dJ/dt \equiv 0$. ∎

The existence of these two integrals imposes geometric restrictions on the solutions. In particular, the following corollaries follow at once from the geometry and the continuation theorem of differential equations.

**Corollary 2.1.** *The functions* $\operatorname{sn}(t, k)$, $\operatorname{cn}(t, k)$, *and* $\operatorname{dn}(t, k)$ *are periodic in* $t$. *In particular,* $\operatorname{sn}(t, k)$, $\operatorname{cn}(t, k)$, *and* $\operatorname{dn}(t, k)$ *are defined and real analytic for all* $t \in \mathbb{R}$.

*Proof.* The values of the integrals on $(\operatorname{sn}(t, k), \operatorname{cn}(t, k), \operatorname{dn}(t, k))$ are $I = J = 1$. The equation $I = 1$ defines a right circular cylinder centered on the $z$ axis, and $J = 1$ defines a right elliptic cylinder centered on the $y$ axis. These two cylinders intersect in two closed curves $\mathcal{C}$ and $\mathcal{C}'$ on which $z > 0$ and $z < 0$, respectively; see Figure 1. The solution $(\operatorname{sn}(t, k), \operatorname{cn}(t, k), \operatorname{dn}(t, k))$ starts in $\mathcal{C}$ and so remains in $\mathcal{C}$ for all $t$. Since $\mathcal{C}$ is bounded, this solution can be continued for all $t \in \mathbb{R}$ by the continuation theorem for differential equations; see [**4**, p. 3], [**6**, pp. 16–17], or [**7**, pp. 12–13]. Since there are no equilibrium points on $\mathcal{C}$, the solution must traverse all of $\mathcal{C}$ and hence is periodic; we investigate the period in Proposition 2.4. ∎

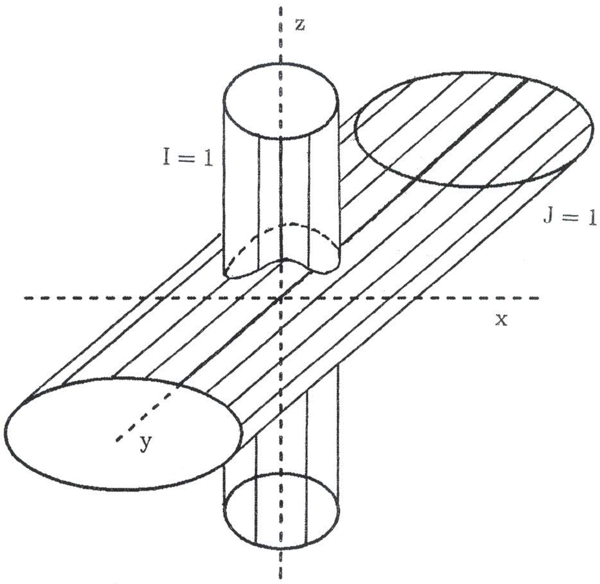


**Figure 1.** The intersection of $I = 1$ and $J = 1$.

**Corollary 2.2.** *For fixed $k$, $0 < k < 1$, and all $t \in \mathbb{R}$ the identities*

$$\operatorname{sn}^2(t,k) + \operatorname{cn}^2(t,k) \equiv 1, \qquad k^2 \operatorname{sn}^2(t,k) + \operatorname{dn}^2(t,k) \equiv 1,$$

*hold, and the inequalities*

$$-1 \le \operatorname{sn}(t,k) \le 1, \qquad -1 \le \operatorname{cn}(t,k) \le 1, \qquad \kappa \le \operatorname{dn}(t,k) \le 1,$$

*are satisfied.*

*Proof.* The identities are restatement of the equations $I = J = 1$. The first identity implies the first two inequalities.

The proof of Corollary 2.1 shows that $\operatorname{dn}(t,k) > 0$. This fact and the second identity imply the last inequality. ∎

**2.3. Symmetries.** Several symmetry properties of the system (1) imply some of the symmetry properties of the Jacobi elliptic functions.

**Proposition 2.3.** *If $(x(t), y(t), z(t))$ is a solution of (1), then so are $(-x(-t), y(-t), z(-t))$, $(x(-t), -y(-t), z(-t))$, and $(x(-t), y(-t), -z(-t))$.*

*Proof.* Let $(\xi(t), \eta(t), \zeta(t)) = (-x(-t), y(-t), z(-t))$; then

$$\begin{aligned} \dot\xi(t) &= \phantom{-}\dot x(-t) = y(-t)z(-t) = \eta(t)\zeta(t), \\ \dot\eta(t) &= -\dot y(-t) = z(-t)x(-t) = -\zeta(t)\xi(t), \\ \dot\zeta(t) &= -\dot z(-t) = k^2 x(-t)y(-t) = -k^2\xi(t)\eta(t). \end{aligned}$$

Thus, $(\xi(t), \eta(t), \zeta(t))$ is a solution also. The other cases follow in the same manner. ∎

Proposition 2.3 says that taking a solution, reversing time, and reflecting through any coordinate plane gives another solution. Such symmetries are known as *time-reversing symmetries.*

The following corollaries illustrate how the uniqueness theorem for differential equations can be used to derive symmetries of the solutions from symmetries of the equations.

**Corollary 2.3.** *For fixed $k$, $0 < k < 1$, $\operatorname{sn}(t,k)$ is an odd function of $t$; $\operatorname{cn}(t,k)$ and $\operatorname{dn}(t,k)$ are even functions of $t$.*

*Proof.* By definition, $(\operatorname{sn}(t,k), \operatorname{cn}(t,k), \operatorname{dn}(t,k))$ is a solution of (1) and hence by Proposition 2.3 so is $(-\operatorname{sn}(-t,k), \operatorname{cn}(-t,k), \operatorname{dn}(-t,k))$, but these two solutions both satisfy the initial condition $(0, 1, 1)$. Thus, the basic uniqueness theorem for ordinary differential equations ensures that they are identical; see [**4**, pp. 1–4], [**6**, pp. 18–24], or [**7**, pp. 31–34]. ∎

Consider the solution $(x(t), y(t), z(t)) = (\operatorname{sn}(t,k), \operatorname{cn}(t,k), \operatorname{dn}(t,k))$ of equation (1) and refer to Figure 1. It starts at $(0, 1, 1)$ and moves into the first octant $(x > 0, y > 0, z > 0)$ where $\operatorname{sn}(t,k)$ increases and $\operatorname{cn}(t,k)$ and $\operatorname{dn}(t,k)$ decrease. Let $K > 0$ be the time that $\operatorname{cn}(t,k)$ takes to decrease to zero, i.e., $\operatorname{cn}(K,k) = 0$ and $\operatorname{cn}(t,k) > 0$ for $0 < t < K$. From Corollary 2.2 we have

$$\begin{aligned}
&\operatorname{sn}(0,k)=0, && \operatorname{sn}(K,k)=1, && 0<\operatorname{sn}(t,k)<1 && \text{for } 0<t<K,\\
&\operatorname{cn}(0,k)=1, && \operatorname{cn}(K,k)=0, && 0<\operatorname{cn}(t,k)<1 && \text{for } 0<t<K,\\
&\operatorname{dn}(0,k)=1, && \operatorname{dn}(K,k)=\kappa, && \kappa<\operatorname{dn}(t,k)<1 && \text{for } 0<t<K.
\end{aligned}\tag{6}$$

**Proposition 2.4.** *As functions of $t$, $\operatorname{sn}(t,k)$ and $\operatorname{dn}(t,k)$ are even about $K$ and $\operatorname{cn}(t,k)$ is odd about $K$, i.e., for fixed $k$, $0<k<1$, and all $t\in\mathbb{R}$*

$$\begin{aligned}
\operatorname{sn}(K+t,k)&\equiv\operatorname{sn}(K-t,k),\\
\operatorname{cn}(K+t,k)&\equiv-\operatorname{cn}(K-t,k),\\
\operatorname{dn}(K+t,k)&\equiv\operatorname{dn}(K-t,k).
\end{aligned}\tag{7}$$

*Thus, $\operatorname{sn}(t,k)$ and $\operatorname{cn}(t,k)$ are $4K$ periodic in $t$ and $\operatorname{dn}(t,k)$ is $2K$ periodic in $t$.*

*Proof.* Since equation (1) is time-independent, $(\operatorname{sn}(K+t,k),\operatorname{cn}(K+t,k),\operatorname{dn}(K+t,k))$ is a solution of (1), and by Proposition 2.3, so is $(\operatorname{sn}(K-t,k),-\operatorname{cn}(K-t,k),\operatorname{dn}(K-t,k))$, but these two solutions both satisfy the initial condition $(1,0,\kappa)$. Thus, the basic uniqueness theorem for ordinary differential equations ensures that they are identical and therefore (7) follows; see [**4**, pp. 1–4], [**6**, pp. 18–24], or [**7**, pp. 31–37].

By (7) and Corollary 2.3, $\operatorname{sn}(t+K,k)\equiv\operatorname{sn}(-t+K,k)\equiv-\operatorname{sn}(t-K,k)$ or $\operatorname{sn}(t+2K,k)\equiv-\operatorname{sn}(t,k)$, from which it follows that $\operatorname{sn}(t,k)$ is $4K$ periodic in $t$. The other cases are similar. ∎

Proposition 2.4 says that $\operatorname{sn}(t,k)$ and $\operatorname{cn}(t,k)$ have the same symmetries with respect to $K$ as $\sin t$ and $\cos t$ have with respect to $\pi/2$.

**2.4. Other differential equation definitions.** The Jacobi elliptic functions satisfy many other important functional equations. Here are some of the classical differential equations that are important in the theory.

**Proposition 2.5.** *The functions $x=\operatorname{sn}(t,k)$, $y=cn(t,k)$, and $z=dn(t,k)$ satisfy the first order equations*

$$\begin{aligned}
\dot{x}^2&=(1-x^2)(1-k^2x^2), && x(0)=0,\ \dot{x}(0)=1,\\
\dot{y}^2&=(1-y^2)(\kappa^2+k^2y^2), && y(0)=1,\ \dot{y}(0)=0,\\
\dot{z}^2&=(1-z^2)(z^2-\kappa^2), && z(0)=1,\ \dot{z}(0)=0.
\end{aligned}\tag{8}$$

*Proof.* From (1) and Corollary 2.2 we have

$$\ddot{x}=\dot{y}z+y\dot{z}=-xz^2-k^2xy^2=-x(1-k^2x^2)-k^2x(1-x^2)=-(1+k^2)x+2k^2x^3.\tag{9}$$

Thus, $\operatorname{sn}(t,k)$ satisfies (9). But (9) has an integral $L=\dot{x}^2+(1+k^2)x^2-k^2x^4$, which is equal to 1 on $\operatorname{sn}(t,k)$ since $\operatorname{sn}(0,k)=0$, $\dot{\operatorname{sn}}(0,k)=1$. Rearranging the equation $L=1$ shows that $\operatorname{sn}(t,k)$ satisfies the first equation in (8). There is a similar argument for the $\operatorname{cn}(t,k)$ and $\operatorname{dn}(t,k)$ equations. ∎

**Corollary 2.4.** $\operatorname{sn}(t,k)$ *is concave down for $0<t<2K$ and concave up for $-2K<t<0$.*

*Proof.* Since $\mathrm{sn}(t,k)$ is positive for $0 < t < 2K$ and $0 < k < 1$, the function $(1+k^2)x - 2k^2x^3$ evaluated at $x = \mathrm{sn}(t,k)$ is positive when $0 < t < 2K$. Hence, (9) ensures that $\ddot{x}$ is negative when $0 < t < 2K$. ■

At this point we have all the basic qualitative features of the Jacobi functions needed to sketch their graphs. Software packages such as Maple and Mathematica have built-in Jacobi functions and nice graphical routines; see Figure 2.

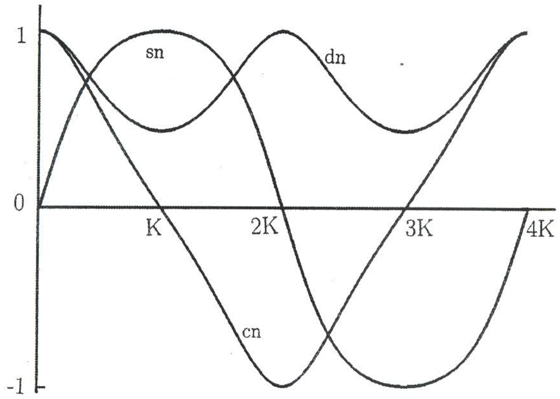


**Figure 2.** The graphs of $\mathrm{sn}(t,k)$, $\mathrm{cn}(t,k)$ and $\mathrm{dn}(t,k)$ for $k = .95$.

**2.5. The integral definition.** The equations in (8) are "solvable up to quadrature". For $0 < t < K$ we have $\dot{x} > 0$ and hence

$$\frac{dx}{dt} = \sqrt{(1-x^2)(1-k^2x^2)}.$$

This is a separable equation, so

$$\int_0^{\mathrm{sn}(t,k)} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = t. \tag{10}$$

This classical definition of $\mathrm{sn}(t,k)$ is found in many texts; it is the natural analog of defining $\sin t$ by

$$t = \int_0^{\sin t} \frac{dx}{\sqrt{1-x^2}}.$$

The integral in (10) converges as the upper limit tends to 1, so

$$K = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}. \tag{11}$$

In this integral make the substitution $x = \sin u$ to get

$$K = \int_0^{\pi/2} \frac{du}{\sqrt{1-k^2\sin^2 u}}. \tag{12}$$

Since the integral (12) doesn't have a singularity, we have:

**Proposition 2.6.** *The integral in (12) defines $K$ as an even, positive real analytic function of $k$ for $-1 < k < 1$. The function $K$ is increasing on $0 < k < 1$, since $dK/dk > 0$ for $0 < k < 1$. Moreover, $K(0) = \pi/2$ and $K \to +\infty$ as $k \to 1-$.*

There are similar integral formulas for cn and dn.

**3. APPLICATIONS.** The references **[2]**, **[3]**, **[5]**, and **[11]** give many applications of Jacobi elliptic functions in mathematics, physics, and engineering. Here are some applications that are appropriate for a course in differential equations.

**3.1. The Pendulum Equation.** The equation for the simple pendulum with all constants set to 1 is

$$\ddot{\theta} + \sin\theta = 0. \tag{13}$$

It admits the energy integral

$$2I = \frac{1}{2}\dot{\theta}^2 + (1 - \cos\theta) = \frac{1}{2}\dot{\theta}^2 + 2\sin^2\frac{\theta}{2}. \tag{14}$$

Set $y = \sin\theta/2$ and $2\dot{y} = \sqrt{1 - y^2}\,\dot{\theta}$ in (14) to get

$$\dot{y}^2 = (I - y^2)(1 - y^2). \tag{15}$$

Case A: $I = 0$ corresponds to the downward equilibrium position; $y(t) \equiv \theta(t) \equiv 0$ is the only solution.

Case B: $0 < I < 1$ corresponds to oscillatory solutions where the pendulum swings back and forth. Set $I = k^2$ and note that $y = k\ \mathrm{sn}(t - \tau, k)$ satisfies (15) for any constant $\tau$, so the solution of the pendulum equation in this case is $\theta(t) = 2\arcsin k\ \mathrm{sn}(t - \tau, k)$, where $I = k^2$.

Case C: $I = 1$ corresponds to the upward equilibrium position and to the solutions that are asymptotic to the upward equilibrium position. $\theta(t) \equiv \pi$ is the upward equilibrium solution. For the asymptotic solutions note that $y = \pm\tanh(t - \tau)$ satisfies (15), so the solution of the pendulum equation is $\theta(t) = \pm 2\arcsin\tanh(t - \tau)$.

Case D: $I > 1$ corresponds to circulating orbits, where the pendulum's energy is high enough to carry the pendulum over the top. Set $I = k^{-2}$ and note that $y = \mathrm{sn}(t/k, k)$ satisfies (15), so the solution of the pendulum equation is $\theta(t) = \pm 2\arcsin\ \mathrm{sn}((t - \tau)/k, k)$. In this case, $\theta$ increases (or decreases) forever, so one must switch branches of the arcsine function so that $\theta$ increases (or decreases) continuously.

**3.2. Elliptic integrals.** A careful reading of the chapter on integration techniques in a standard calculus book shows that if $R$ is a rational function of $\sqrt{X}$ and $x$ and if $X$ is linear or quadratic in $x$, then any integral of the form

$$\int R(x, \sqrt{X})\, dx$$

can be integrated using elementary functions. By completing the square, using trigonometric substitutions, partial fractions, etc., one expresses the integral in terms of the trigonometric functions, logarithms, and exponentials.

The theory of elliptic integrals investigates integrals of the same form, where $X$ is now a cubic or quartic in $x$. There are reduction methods to reduce any integral of this form to either elementary integrals or to what are known as elliptic integrals of the first, second, and third kinds. Jacobi elliptic functions can be used to evaluate any integral of the first kind, i.e., any integral of the form

$$\int \frac{dx}{\sqrt{X}}. \tag{16}$$

A complete discussion with all the degenerate cases can be found in [**2**, pp. 4–15], [**3**, pp. 86–98], and [**5**, pp. 31–42]. Since the development is lengthy, we give only a brief hint of the theory.

When $X$ is a quartic, one can find constants $p$ and $q$ such that the change of variables $x = (p+qy)/(1+y)$, $dx = (q-p)dy/(1+y)^2$ reduces (16) to

$$I = (q-p)\int \frac{dy}{\sqrt{Y}},$$

where $Y$ is a quadratic in $y^2$. When $X$ is a cubic and $a$ is a real root of $X$, the change of variables $x = y^2 + a$, $dx = 2y\,dy$ effects a similar reduction.

In both cases the problem is reduced to integrating an integral of the first kind (16) where $X$ is a quadratic in $x^2$ that can be factored. Then elementary tricks reduce the integral to a standard form such as (10). Here is an example. Let $a$ and $b$ be constants with $0 < a < b$. Then

$$\begin{aligned} \int_0^x \frac{du}{\sqrt{(a^2-u^2)(b^2-u^2)}} &= \int_0^{x/a} \frac{dv}{\sqrt{(1-v^2)(b^2-a^2v^2)}} && (u = av) \\ &= \frac{1}{b}\int_0^{x/a} \frac{dv}{\sqrt{(1-v^2)(1-k^2v^2)}} && (k = a/b) \quad (17) \\ &= \frac{1}{b}\,\mathrm{sn}^{-1}(x/a, a/b) \end{aligned}$$

**3.3. Systems with quadratic or cubic forces.** If $f(x)$ is either a quadratic or cubic polynomial in $x$, then any differential equation of the form

$$\ddot{x} + f(x) = 0 \tag{18}$$

is solvable in terms of the Jacobi elliptic functions.

Equation (18) admits the integral

$$I = \dot{x}^2 + X(x), \quad \text{where} \quad X(x) = 2\int f(x)\,dx \tag{19}$$

as a constant of motion, so the phase portrait can be obtained by plotting the level lines of $I$. By setting $I = c$, a constant, (19) becomes a separable equation. Thus,

$$\int \frac{dx}{\sqrt{c - X(x)}} = t, \tag{20}$$

and since $f(x)$ is either a quadratic or a cubic, $c - X(x)$ is either a cubic or a quartic. Therefore (18) can be integrated by the methods outlined in the previous section.

For example, consider the undamped Duffing equation

$$\ddot{x} + x - 2x^3 = 0$$

with integral $I = \dot{x}^2 + x^2 - x^4$. Let $I = c, 0 < c < 1/4$, and seek a solution satisfying $x(0) = 0, \dot{x} > 0$ so that (20) becomes

$$\int_0^x \frac{dx}{\sqrt{x^4 - x^2 + c}} = t.$$

Since $0 < c < 1/4$, the polynomial factors to give $x^4 - x^2 + c = (a^2 - x^2)(b^2 - x^2)$ with $0 < a < b$. Thus, (17) shows that $x(t) = a\ \mathrm{sn}(bt, a/b)$.

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