On Contact Transformations

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It is well known that a transformation of the form

$$p_i = \frac{\partial W}{\partial q_i}(q,P), \quad Q_i = \frac{\partial W}{\partial p_i}(q,P) \qquad (1)$$

defines a contact transformation from the q,p variables to the Q,P variables where q,p,Q,P are n vectors, W is a scalar function with continuous second partial derivatives with respect to all arguments, and subscripts denote components of the vectors.[1, 2] It is not always true that any contact transformation can be written in the form (1) or even in one of the other three common variations of (1). This fact is pointed out in Refs. 2 and 3, and the author recommends Ref. 2 as a careful and readable source on contact transformations (see in particular, pp. 69 and 70 of Ref. 2). This note shows, however, that *any contact transformation can be written as a composition of a linear orthogonal contact transformation and a contact transformation of the form* (1). That is to say, given any contact transformation one can first make a change of variables that is linear, orthogonal and preserves Hamiltonian form and then write the transformation in the form (1). The above is to be taken as a local statement, that is, the above statement holds only in a sufficiently small neighborhood of a point. Also, we assume that all functions are sufficiently differentiable that the indicated derivatives are continuous and that the implicit function theorem can be applied. The assumption that all functions considered have continuous second partial derivatives with respect to all arguments will suffice.

To avoid confusion, a contact transformation is taken in the sense of Whittaker, p. 293.[1] That is:

Definition: A transformation

$$\mathcal{F}\colon Q = \varphi(q,p), \quad P = \psi(q,p), \qquad (2)$$

where q,p,Q,P are n vectors and ψ and φ are n-vector-valued functions of q and p is called a contact transformation if there exists a scalar-valued function $S(q,p)$ such that

$$dS(q,p) = \sum_{i=1}^{n} \{p_i dq_i + \varphi_i(q,p)\, d\psi_i(q,p)\}. \qquad (3)$$

Observe that (3) is often written

$$dS = \sum_{i=1}^{n} \{p_i dq_i + Q_i dP_i\},$$

and that this short notation is the cause of some of the confusion in the literature. The equality (3) states that S must be considered as a function of p and q only. Indeed, the whole question of when a contact transformation (2) can be written in the form (1) rests on the question of when can S be written as a function of q,P. If the second equation in (2) can be solved for p in terms of P and q and the result substituted into S, we would have the desired function W. But when can we solve the second equation in (2) for p in terms of q and for P? If the sub-Jacobian $\det\left\{\frac{\partial \psi_i}{\partial p_j}\right\}$ is nonzero, then we can solve this equation, but there is no reason to suppose that it is nonzero. At this point a result in Ref. 3 can be used to straighten things out.

The formal proof is as follows. Let (2) or $\mathcal{F}$ be a given contact transformation. Without loss of generality, we can assume that $\mathcal{F}$ takes the origin into the origin since otherwise we would shift the origin by a translation. Let T be the Jacobian matrix of $\mathcal{F}$ evaluated at the origin, i.e.,

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where

$$A = \left\{\frac{\partial \varphi_i}{\partial q_j}(0,0)\right\}, \quad B = \left\{\frac{\partial \varphi_i}{\partial p_j}(0,0)\right\}, \quad C = \left\{\frac{\partial \psi_i}{\partial q_j}(0,0)\right\}$$

and

$$D = \left\{\frac{\partial \psi_i}{\partial p_j}(0,0)\right\}.$$

Now, by a result in Ref. 3 (p. 44), there exist nonsingular contact matrices O and R where O is orthogonal and R is positive definite symmetric such that $T = RO$. This result for contact matrices is the analog of the well-known result in three dimensions that says that any matrix of a linear transformation is the product of a pure rotation (or rotation and reflection) and a pure dilation. It should be remarked that in Ref. 3, as in many other references, a contact matrix is called "symplectic" and is sometimes given a different but equivalent definition.[2]

Let O be the transformation whose representation is the matrix 0, Define a new transformation $\mathcal{G}$ by $\mathcal{G} = \mathcal{F}\circ O^{-1}$; and so $\mathcal{F} = \mathcal{G}\circ O$. Observe that we have "factored" the transformation $\mathcal{F}$ into two operations: first apply O and then $\mathcal{G} = \mathcal{F}\circ O^{-1}$. Another way of looking at $\mathcal{G}$ is that we have changed coordinates by the linear transformation O and now $\mathcal{F}$ has the form $\mathcal{G}$ in the new coordinates. We now want to show that $\mathcal{G}$ can be written in the form (1).

$\mathcal{G}$ is a contact transformation, since it is the composition of two contact transformations and moreover, its Jacobian matrix at the origin is $TO^{-1} = (RO)O^{-1} = R$. Thus, if $\mathcal{G}$ is given by $Q = a(q',p')$, $P = b(q',p')$ and

$$R = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix},$$

where

$$A' = \left\{\frac{\partial a_i}{\partial q_j'}(0,0)\right\}, \text{ etc.}$$

Now R is positive definite and symmetric, and so, by Sylvester's criterion, [4,5] each principal subdeterminant of R is positive and therefore, in particular,

$$D' = \left\{\frac{\partial b_i}{\partial p_i'}(0,0)\right\} \text{ is nonsingular.}$$

Thus, we can solve the equation $P = b(q',p')$ for p' to obtain $p' = h(q',P)$.

Since $\mathcal{G}$ is a contact transformation there exists a generating function $S'(q',p')$ such that

$$dS'(q',p') = \sum_{i=1}^{n} \{p_i'dq_i' + b_i(q',p')\, da_i(q',p')\}. \quad (4)$$

Let $W(q',P) = S'[q',h(q',P)]$. Now,

$$dW(q',P) = \sum_{i=1}^{n} \left\{ \frac{\partial W}{\partial q'} \; dq_i' + \frac{\partial W}{\partial P_i} dP_i \right\} \quad (5)$$

but $dW = dS$ at corresponding points; and so

$$dW(q',P) = \sum_{i=1}^{n} \{p_i'dq_i' + b_i(q',p')dP_i\}, \quad (6)$$

where in (6) $p' = h(q',P)$.

Now, since

$$dP_i = \sum_{j=1}^{n} \left\{ \frac{\partial b_i}{\partial q_j'} dq_j' + \frac{\partial b_i}{\partial p_j'} dp_j' \right\} \text{ and } \left\{ \frac{\partial b_i}{\partial p_j} \right\}$$

is nonsingular, the differentials $dq_1', \cdots, dq_n', dP_1, \cdots, dP_n$ are linearly independent and so we can equate coefficients in (5) and (6) to obtain

$$p_i' = \frac{\partial W}{\partial q_i'}(q',P) \text{ and } Q_i = \frac{\partial W}{\partial P_i}(q',P). \quad (7)$$

Therefore, $\mathcal{G}$ is of the form (1).

Observe that we can obtain one of the other common variations of (1) when any one of the other sub-Jacobian matrices is nonsingular. The procedure we have used gives that A' is nonsingular, so this gives one variant. By changing variables again with the linear orthogonal contact matrix

$$\begin{bmatrix} O & I \\ -I & O \end{bmatrix},$$

then the Jacobian of the new $\mathcal{G}$ is of the form

$$\begin{bmatrix} -B' & A' \\ -D' & C' \end{bmatrix}.$$

Therefore, now the upper right and lower left sub-Jacobian matrices are nonsingular and, by the same procedure, you get the other two variants.

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4 F. R. Gantmacher, *The Theory of Matrices* (Chelsea Publishing Company, New York, 1959), Vol. 1, p. 306.

5 S. Perlis, *Theory of Matrices* (Addison–Wesley Publ. Co., Reading, Mass., 1958), p. 94.

On the Equivalence of Truncated Ring Pendula

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A standard experiment in the elementary laboratory[1] introduces students to the ideas and methods of experimental induction by having them obtain empirically, without previous derivation, the functional dependence of the period of a ring pendulum on its diameter. Students carrying out this experiment are surprised and intrigued by the discovery that the equivalent simple pendulum has a length equal to the ring diameter. Slightly more sophisticated and equally surprising is the discovery that the period of the pendulum is not altered by removing from the ring a segment symmetrical about its vertical axis. This remarkable fact, which is known to many teachers of physics,[2] suggests the following extension to cases where the radial thickness of the ring is not negligibly small; in the more general case also, the removal of a symmetrical sector has no effect on the period, and the equivalent simple pendulum has a length equal to twice the radius of gyration of the ring about its geometrical center.

Consider Fig. 1, which is a drawing of an annular ring of inner radius R_1 and outer radius R_2. The ring has been truncated by removing a sector of angular width 2θ. In addition, a small hole has been drilled on the line MN so that, when the ring is suspended from this hole, the center C of the ring is directly below the point of support. Let the ring swing as a physical pendulum about an axis through a point P at the top of the hole, with the axis perpendicular to the face of the ring, and the distance PC to the geometrical center made equal to the radius of gyration about

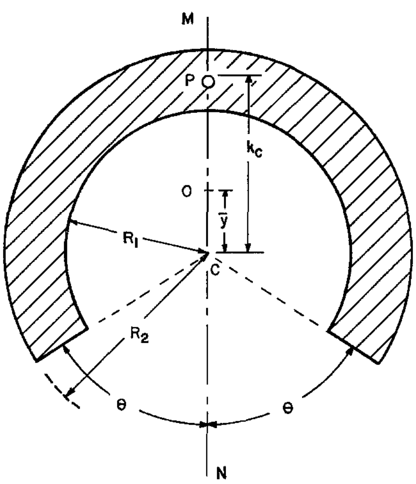

Fig. 1. Drawing of a truncated annular ring to be used as physical pendulum when suspended from the point P (The symbols are defined in the text.)