PERIODIC ORBITS AND SOLENOIDS IN GENERIC HAMILTONIAN DYNAMICAL SYSTEMS

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1. Minimal Sets of Hamiltonian Dynamics: Points, Circles, Tori, and Solenoids. The recurrent trajectories of conservative dynamical systems have been investigated intensively since the age of Lagrange and Hamilton [12, 18]. In the qualitative theory of dynamical systems [2, 13] recurrent motions are often found by locating a minimal set, that is, a compact invariant set within which every trajectory is relatively dense and hence recurrent. Classical examples of minimal sets for Hamiltonian dynamical systems are points (critical or equilibrium points), topological circles (periodic orbits), and tori filled by almost periodic trajectories. In fact, points, circles and tori are the only types of minimal sets for solvable problems of Hamiltonian dynamics, which are completely integrable as decoupled 1-dimensional oscillators. However, not all these circles and tori of a solvable problem will persist as periodic orbits or almost periodic trajectories under Hamiltonian perturbations of a generic nonsymmetrical nature; and furthermore more complicated types of minimal sets will usually appear containing new sorts of recurrent trajectories.

The principal result of the present investigation is that Hamiltonian dynamical systems, under appropriate generic conditions, necessarily have minimal sets that are topological solenoids of every possible type, and thus these minimal sets are not manifolds.

The local behavior of a Hamiltonian dynamical system is described by Hamiltonian differential equations

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x^i} \qquad i = 1, \ldots, n$$

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or the vector differential system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = J \begin{pmatrix} H_x \\ H_y \end{pmatrix}$$

in the real number space $\mathbf{R}^{2n}$. Here $H(x^1, \ldots, x^n, y_1, \ldots, y_n)$ is the given real Hamiltonian function of class C^{k+1} for $k = 1, 2, \ldots, \infty$, with gradient $dH = (H_x, H_y)$ transposed as a column vector, and

$$J = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$$

is a standard skew-symmetric $2n \times 2n$ matrix displaying the unit matrix E_n. The linear symplectic space $\mathbf{R}^{2n}$ is usually denoted $\mathbf{R}^n \oplus \mathbf{R}^n$ to emphasise the use of the canonical coordinates (x, y).

We shall present our researches in terms of the global theory of generic Hamiltonian systems on a symplectic manifold M, as described below and in section two later (see [1] and the Memoir [9] for the basic concepts and further references.) We recall that a symplectic manifold M is a differentiable $2n$-manifold (a connected separable, metrizable manifold without boundary, and with a distinguished maximal atlas of C^∞ local coordinates or charts) together with a prescribed symplectic form Ω(*a* C^∞ 2-form that is everywhere closed and nonsingular on M). The Theorem of Darboux asserts that M is covered by special local charts, called canonical coordinates $(x^1, \ldots, x^n, x^{n+1}, \ldots, x^{2n})$ in which

$$\Omega = \sum_{j=1}^{n} dx^j \wedge dx^{n+j}.$$

In such a canonical coordinate chart, usually denoted $(x^1, \ldots, x^n, y_1, \ldots, y_n)$, the components of the tensor Ω are just the constant matrix

$$J = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}.$$

A canonical map or symplectomorphism between sympectic manifolds is a diffeomorphism that preserves the symplectic form. In this case the Jacobian matrix T, computed in terms of canonical coordinates, satisfies the identity $TJT' = J$ at each point. That is, T belongs to the symplectic linear group $Sp(2n, \mathbf{R})$, which is a Lie group whose Lie algebra consists of all Hamiltonian matrices A with $AJ + JA' = 0$.

The symplectic form Ω on M defines an alternating bilinear product $\{u, v\} = \Omega(u, v)$ for vectors u, v in any tangent space T_PM at $P \in M$. In this way T_PM becomes a symplectic linear space isomorphic to $\mathbf{R}^n \oplus \mathbf{R}^n$, and this isomorphism is displayed explicitly in terms of canonical coordinates around P wherein $\Omega(u, v) = u'Jv$.

By means of the symplectic form Ω we construct a duality between contravariant and covariant vectors on M. Namely, if σ is a 1-form at $P \in M$, then the corresponding tangent vector is denoted by $\sigma^\#$ (and also $\sigma = (\sigma^\#)_\flat$) where $\sigma(v) = \{\sigma^\#, v\} = \Omega(\sigma^\#, v)$ for each $v \in T_P$.

A Hamiltonian H is a real C^{k+1}-function, for $k = 1, 2, \ldots, \infty$, on the symplectic manifold M. The gradient dH has the dual $dH^\#$ or X_H which is the corresponding Hamiltonian vector field in class C^k on M. In any canonical coordinate chart $(x^1, \ldots, x^n, y_1, \ldots, y_n)$ on M the tangent vector field $dH^\#$ is given by the local Hamiltonian differential system

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial y_i}, \qquad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x^i} \qquad i = 1, \ldots, n.$$

The trajectories (solution or integral curves) of such a Hamiltonian vector field, constructed always from a single-valued global Hamiltonian function, define a (local) differentiable flow on M. In the most important cases we treat, where M is compact or else the cotangent bundle of a compact n-manifold, this flow is continued for all times $t \in \mathbf{R}$. It is the trajectories and the flow of such a Hamiltonian dynamical system on the symplectic manifold M that are investigated here.

Each Hamiltonian flow is known to yield a homomorphism of the additive group $\mathbf{R}$ into the group Symp(M) of all C^k-symplectomorphisms of M. Furthermore, since

$$\frac{dH}{dt} = \frac{\partial H}{\partial x^i}\dot{x}^i + \frac{\partial H}{\partial y_i}\dot{y}^i \equiv 0$$

as computed along any trajectory of $dH^\#$ in terms of canonical coordinates, the energy H is conserved; that is, each energy level $H = h$ is an invariant set for the Hamiltonian flow $dH^{\#}$.

While we shall mainly restrict our attention to the case when M is compact (for instance, the $2n$-torus $T^n \oplus T^n$), most of our technical analysis will take place within one local canonical chart (x, y). Thus much of our study refers to the behavior of a Hamiltonian differential system near the origin in the linear symplectic space $\mathbf{R}^n \oplus \mathbf{R}^n$.

Example 1. Let H be a Hamiltonian on M with a critical point at Q_0, say $Q_0 = (0, 0)$ in canonical coordinates (x, y). Then $dH = 0$ at Q_0 so, ignoring the inessential constant $H(0, 0)$,

$$H(x, y) = \frac{1}{2}(x, y)S\begin{pmatrix} x \\ y \end{pmatrix} + \cdots$$

for some real symmetric matrix $S = S'$. Also the Hamiltonian differential system $dH^{\#}$ has a critical point at Q_0 and near this origin it has the format

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = JS\begin{pmatrix} x \\ y \end{pmatrix} + \cdots$$

where $A = JS$ is a Hamiltonian matrix in the Lie algebra $sp(2n, \mathbf{R})$.

The eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n, -\lambda_1, -\lambda_2, \ldots, -\lambda_n)$ of the matrix A are invariants of the Hamiltonian system about the critical point Q_0, and are independent of the choice of the canonical chart. The critical point Q_0 is called degenerate in case some eigenvalue is zero, that is, A or S is singular.

If all these eigenvalues are nonzero pure-imaginary numbers, then Q_0 is an elliptic critical point for the Hamiltonian system $dH^{\#}$. This is certainly the case when S is strictly positive-definite, since then the point Q_0 is Liapunov stable for the Hamiltonian flow.

Example 2. Let H be a Hamiltonian on M with a periodic orbit γ. The behavior of the solutions of $dH^{\#}$ in a tubular neighborhood of γ can be studied by means of the Poincaré map P of a transversal $(2n - 1)$-section Σ through any point Q_0 on γ. Here P maps Σ (or some neighborhood of $Q_0 \in \Sigma$) into Σ by following the trajectories of $dH^{\#}$ once around the tube encircling γ. Also P restricts to a map P_h on each energy level $H = h$ or $\Sigma(h)$ in Σ.

The eigenvalues $(\mu_2, \mu_3, \ldots, \mu_n, \mu_2^{-1}, \mu_3^{-1}, \ldots, \mu_n^{-1})$ of the Jacobian matrix of the Poincaré map P_h at $Q_0 \in \Sigma$ are the nontrivial characteristic multipliers of the periodic orbit γ, and these are independent of the choice of Σ and the local chart. The periodic orbit is called degenerate in case some (nontrivial) characteristic multiplier has the value $+1$. If γ is nondegenerate, then γ lies in a local band or 2-cylinder filled by periodic orbits $\gamma(h)$ of $dH^{\#}$, whose (least positive) period varies smoothly with the parameter h which is the energy level. Furthermore, if all these characteristic multipliers μ_j of γ are distinct for $j = 2, \ldots, 2n$, then the corresponding multipliers $\mu_j(h)$ of the $\gamma(h)$ vary smoothly with the energy h.

If γ is nondegenerate and every (nontrivial) characteristic multiplier has a modulus of one, then γ is called an elliptic periodic orbit. In this case we can define the characteristic frequences $\omega_j \pmod 1$ by $\mu_j = e^{2\pi i \omega_j}$.

Remarks. A necessary condition that a nondegenerate critical point or periodic orbit be Liapunov stable is that it be elliptic. Elliptic periodic orbits will play a central role in our theory. By a theorem of Liapunov, elliptic orbits necessarily exist near an elliptic critical point, at least under generic conditions as discussed later. Moreover, within a tubular neighborhood of such an elliptic orbit there will be shown to exist other long-period elliptic orbits which encircle the tube a large number of times before completing their periods. We shall prove below that generic Hamiltonians admit sequences of elliptic orbits, with carefully selected long-period encirclings, and these converge to minimal sets of specified solenoidal types.

It is convenient now to define solenoids as topological spaces, to list some of their important properties, and to indicate a standard type of minimal flow on each such solenoid. Consider first a fixed positive prime p, and then define a p-adic solenoid Σ_p by the following construction, as an intersection of a nested sequence of solid tori in $\mathbf{R}^3$. After this, a more general type of solenoid Σ_a will be defined for each sequence $a = (a_0, a_1, a_2, \ldots)$ for integers $a_j \geq 2$ (the p-adic solenoid corresponds to the choice $a_j = p$ for all $j = 0, 1, 2, \ldots$).

Let T_0 be a solid torus in a standard embedding in $\mathbf{R}^3$. Let T_1 be a solid torus, lying within the interior of T_0 and longitudinally encircling it p-times. Then let T_2 be a solid torus, lying within the interior of T_1 and encircling it p-times (hence encircling the torus T_0 p^2-times). Continue in this fashion to define T_{j+1} encircling the torus T_j p-times, and take the meridianal diameters of T_j tending to zero as j increases. Then the solenoid Σ_p is defined as the intersection $\Sigma_p = \bigcap^{\infty}_{j=0} T_j$, which is a nonempty compact subset of $\mathbf{R}^3$. It is known that distinct primes give rise to topologically distinct solenoids [4 p. 122].

A more general type of solenoid Σ_a is specified for each sequence $a = (a_0, a_1, a_2, \ldots)$ for integers $a_j \geq 2$. To obtain Σ_a proceed as in the above construction (where each $a_j = p$), except that T_{j+1} encircles the torus T_j a_j-times. A more concise, but entirely equivalent, definition of Σ_a can be made by emphasizing the central longitudinal circle of each T_j, rather than the toroidal tube itself.

For this approach consider an infinite sequence of maps of the circle S^1 into itself:

$$S^1 \overset{h_0}{\leftarrow} S^1 \overset{h_1}{\leftarrow} S^1 \overset{h_2}{\leftarrow} S^1 \overset{h_3}{\leftarrow} \cdots$$

where $h_j : z \to z^{a_j}$ for $j = 0, 1, 2, \ldots$, and S^1 is taken to be the unit circle in the complex number plane. Then the inverse (or projective) limit of this mapping system is the solenoid

$$\Sigma_a = \lim_{\leftarrow} \{S^1, h_j\}.$$

More explicitly, the solenoid Σ_a is the subset of the denumerable topological product $S^1 \times S^1 \times S^1 \times \cdots$ consisting of all sequences $(z_0, z_1, z_2, \ldots)$ for which $z_j = h_j(z_{j+1})$ for $j = 0, 1, 2, 3, \ldots$. It is well known [4 p. 109, 5] that each such solenoid Σ_a is a compact metric space which is connected and 1-dimensional (so Σ_a is a Klosed Kurve in the sense of Menger). However, Σ_a is not locally connected and hence Σ_a cannot be a topological manifold.

Since each of the maps h_j is a group homomorphism of S^1, the projective limit Σ_a is also a compact abelian topological group. The solenoids Σ_a for $a = (a_0, a_1, a_2, \ldots)$ and Σ_b for $b = (b_0, b_1, b_2, \ldots)$ are topologically isomorphic groups provided [5 p. 114, 404, 417]:

p^r divides some product $(a_0 a_1 a_2 \cdots a_k)$ if and only if
p^r divides some product $(b_0 b_1 b_2 \cdots b_l)$,
for every choice of the positive prime power p^r.

While there exist uncountably many topological types of such solenoids, the above criterion shows that we have considerable leeway in the selection of the integer sequence $a = (a_0, a_1, a_2, \ldots)$ during the construction of any specified solenoid.

From the viewpoint of the theory of topological groups we can characterize these solenoids as the most general compact, connected, 1-dimensional, torsionless abelian groups [5 p. 418]. The significance of this group-theoretic description, within the framework of dynamical systems theory, is that a minimal set on which the flow is Bohr almost periodic is necessarily the space of a compact, connected, abelian group [13 p. 394]. The solenoids arising in our analysis of generic Hamiltonian systems carry Bohr almost periodic flows, after some possible modifications in the trajectory speed. In fact, these minimal flows can be described in terms of the geometric definition of solenoids, as intersections of encircling nested solid tori in $\mathbf{R}^3$, with constant longitudinal angular trajectory speed, see [13, p. 392].

For greater precision we now define such a standard minimal flow ϕ_t on a solenoid Σ_a by means of rotations on each of the circular components $S^1 = \{z \in \mathbf{C} : |z| = 1\}$. Namely, for $t \in \mathbf{R}$ let

$$\phi_t : (z_0, z_1, z_2, \ldots) \to (e^{it}z_0, e^{it/a_0}z_1, e^{it/a_0a_1}z_2, \ldots).$$

This flow is easily seen to be minimal on Σ_a.

Each solenoidal minimal flow, which we find for a generic Hamiltonian C^k-vector field $dH^\#$ on M, is conjugate to one of the above flows on some Σ_a, after multiplication of $dH^\#$ by a positive C^k-function on M in order to modify the time-parametrization of the trajectories.

We are now able to state our Principal Theorem as proved in Section 4 below. The clarification of the related concepts, especially concerning the generic subsets of the Baire space $\mathfrak{H}^k$ of all Hamiltonian dynamical systems (of class C^k for $k \geq 4$) on the compact symplectic manifold M (of dimension $2n \geq 4$), will be given in Section 2.

PRINCIPAL THEOREM. *Let $\mathfrak{H}^k$ be the space of Hamiltonian dynamical systems on the compact symplectic manifold M. Then there exists a generic set $\mathfrak{M}_\Sigma \subset \mathfrak{H}^k$ such that:*

> *for each Hamiltonian system $dH^\# \in \mathfrak{M}_\Sigma$, and for each solenoid Σ_a, there exists a minimal set for the flow of $dH^\#$ that is homeomorphic to Σ_a.*

The generic set $\mathfrak{M}_\Sigma$ will be defined as an intersection $\mathfrak{M}_\Sigma = \mathfrak{R} \cap \mathfrak{S}_0 \cap \mathfrak{S}_1 \cap \mathfrak{S}_2 \cap \mathfrak{S}_3$ of five explicit generic sets in Section 3 below. Roughly speaking these generic sets consist of Hamiltonian systems with the following properties:

$\mathfrak{R}$ ~ almost all (excepting countably many) periodic orbits are nondegenerate, and have distinct characteristic multipliers.

$\mathfrak{S}_0$ ~ all critical points are generic; that is, each has eigenvalues $(\lambda_1, \ldots, \lambda_n)$ that are rationally independent.

$\mathfrak{S}_1$ ~ all periodic orbits have at most one characteristic multiplier $(\mu_2, \ldots, \mu_n)$ that is a root of unity.

$\mathfrak{S}_2$ ~ almost all periodic orbits have characteristic multipliers that change (non-constant) with the energy level h within the corresponding local 2-cylinder.

$\mathfrak{S}_3$ ~ almost all elliptic periodic orbits have a strictly nonlinear Poincaré map; that is, elliptic orbits have a nonzero "twist coefficient."

Condition $\mathfrak{R}$ was studied by Robinson [14, 15]; $\mathfrak{S}_0$ is essentially contained in an earlier Memoir of the authors [9]; and $\mathfrak{S}_1$, $\mathfrak{S}_2$, $\mathfrak{S}_3$ are now analyzed by a method due to Takens [16].

Physical motivations. A few remarks on familiar physical models might be useful as a guide in the subsequent mathematical analysis.

Within dynamical astronomy the rotation of the Earth is determined by Hamiltonian differential equations like those for a spinning top. The Earth rotates on its axis in inertial space once every 24 (sidereal) hours. However, this axis through the Earth is not precisely fixed in direction but itself turns around a small circle, with a radius of a few meters at the North Pole, once every 14 months, thus causing the Chandler Wobble. Even the mean axis is not stationary but rotates around a larger circle once every 25,000 years, thus accounting for the Precession of the Equinoxes.

From the viewpoint of physics and astronomy this sequence of higher order nutational and precessional oscillations is caused by the nonhomogeneity and asymmetry of the Earth under the gravitational forces of the Solar System. Thus the lack of symmetry of this Hamiltonian dynamical problem leads to the existence of higher order periodicities that may accumulate in a complicated type of recurrent motion.

Our theory shows that such behavior of Hamiltonian dynamical systems is typical or generic.

In order to appreciate the method of proof of our Principal Theorem, we briefly consider here another physical example described by a particle sliding near the bottom of a smooth frictionless paraboloidal-like surface, under the downward force of gravity. The surface is (locally) the 2-dimensional position manifold, and the usual momentum phase-space is a symplectic 4-manifold. The Hamiltonian function is the total mechanical energy of any particle as it slides around inside this cup-like surface.

The bottom point of this cup, for zero velocity, is an elliptic critical point for this Hamiltonian system, and is a generic elliptic point in case the two principal radii of curvature of the cup are suitably incommensurable. By the Theorem of Liapunov there are families of periodic oscillations for the particle sliding along each of the principle-curvature sections, and these oscillations are parametrized by the energy, or equivalently by the amplitude. Take one such oscillation from one of these families, and then perturb the motion of the particle by a slight transverse velocity so that it vibrates slightly back and forth across the chosen principal-curvature section. If the original oscillation has a "nonzero twist coefficient," then the period of the transverse vibration can be controlled by the magnitude of the transverse nudge. In this case the two periods can be interrelated so as to produce a new long-period oscillation. In the 4-manifold this long-period oscillation winds several times around a tubular neighborhood of the original oscillation.

With a further sequence of carefully controlled perturbations we obtain long-period orbits that accumulate towards a solenoidal minimal set in the symplectic manifold.

In concluding this introduction we observe that it might seem strange that such a pathological "Klosed Kurve" as a solenoid can be of importance in the theory of smooth conservative flows. Yet examples of such solenoids were discovered decades ago by Morse [11] in his studies of the geometry of geodesic flows, even for analytic systems. Also Birkhoff [2 pp. 218–220] indicated that solenoidal minimal sets are likely structures near an elliptic periodic orbits of any Hamiltonian system of two degrees of freedom.

2. Generic Hamiltonian Systems on Symplectic Manifolds. In this section we present the concepts of generic Hamiltonian differential systems, and exhibit a method of specifying generic classes of Hamiltonian systems by imposing conditions on their periodic orbits. For these purposes we shall first topologize the set $\mathfrak{H}^k$ of Hamiltonian systems as a Baire space, and then we discuss questions related to perturbation and transversality theory. Our main objective in this section will be the explanation of a proposition due to Takens, slightly modified for our usage, and the development of two corollaries that bring this result directly into the domain of our application.

Let $\mathfrak{C}^{k+1}$ be the set of all real functions of class C^{k+1}, for some $k = 1, 2, \ldots, \infty$, on the symplectic C^∞-manifold M of dimension $2n \geq 4$. Each $H \in \mathfrak{C}^{k+1}$ is a Hamiltonian function on M, and the corresponding Hamiltonian vector field $dH^\#$ is of class C^k. Let $\mathfrak{H}^k$ be the set of all such Hamiltonian C^k-vector fields on M, each obtained from a single-valued global Hamiltonian function in $\mathfrak{C}^{k+1}$.

There is a surjective projection onto $\mathfrak{H}^k$

$$\pi : \mathfrak{C}^{k+1} \to \mathfrak{H}^k : H \to dH^\#.$$

The inverse image $\pi^{-1}(dH^\#)$ of a Hamiltonian vector field is a class of Hamiltonians $\{H + c\}$, differing from H by an additive constant. Such an equivalence class $\{H + c\}$ is often called a "normalized Hamiltonian" since it can be defined by a representative function $H - H(Q_0)$ "normalized to zero" at a specified point $Q_0 \in M$. Each vector field $dH^\# \in \mathfrak{H}^k$ corresponds to exactly one normalized Hamiltonian class $\{H + c\}$ in $\mathfrak{C}^{k+1}$, by virtue of the bijection induced by the projection π. Thus we occasionally treat elements of $\mathfrak{H}^k$ as such normalized Hamiltonians.

We impose the Whitney C^{k+1}-topology (stronger than the corresponding compact-open topology) on the space $\mathfrak{C}^{k+1}$. We define a neighborhood $\mathfrak{U}$ of $H \in \mathfrak{C}^{k+1}$ by means of a choice of any finite number of real continuous functions $\epsilon_0(x)$, $\epsilon_1(x)$, $\ldots$, $\epsilon_r(x)$, for $r \leq k + 1$, positive everywhere on M. Then $F \in \mathfrak{C}^{k+1}$ lies in $\mathfrak{U}$ provided the pointwise inequalities hold on M,

$$|F - G| < \epsilon_0, \qquad |D(F - H)| < \epsilon_1, \ldots, |D^r(F - H)| < \epsilon_r.$$

Here D^s is any covariant derivative of total order s, and these derivatives and their norms are computed relative to some fixed Riemann C^∞-metric on M. With this neighborhood base $\mathfrak{C}^{k+1}$ becomes a Hausdorff topological space, and the topology is independent of the choice of the auxilliary Riemann metric. Moreover, $\mathfrak{C}^{k+1}$ is a Baire space having the Baire category property, namely, every residual set (a countable intersection of open and dense subsets) is dense in $\mathfrak{C}^{k+1}$.

The only subsets of $\mathfrak{C}^{k+1}$ that we shall encounter will be unions of normalized Hamiltonians, that is, they will be saturated under the equivalence relation of "having the same projection image by π." We note that the saturation $\pi^{-1}(\pi(\mathfrak{O}))$ of an open subset $\mathfrak{O} \subset \mathfrak{C}^{k+1}$ is still open. These considerations motivate us to endow $\mathfrak{H}^k$ with the weakest topology so that $\pi:\mathfrak{C}^{k+1} \to \mathfrak{H}^k$ is an open map. In this case a set of Hamiltonian vector fields is open in $\mathfrak{H}^k$ if and only if the corresponding collection of normalized Hamiltonians constitutes an open set of $\mathfrak{C}^{k+1}$. We easily conclude that there exists a natural homeomorphism

$$\mathfrak{C}^{k+1} \approx \mathfrak{H}^k \times \mathbf{R}:H \leftrightarrow (dH^{\#}, H(Q_0)),$$

and furthermore $\mathfrak{H}^k$ is also a Baire space.

In the important special case where M is compact, both $\mathfrak{C}^{k+1}$ and $\mathfrak{H}^k$ are complete separable metric spaces. In this case we can metrize $\mathfrak{H}^k$ according to the distance formula:

$$d_k(dF^{\#}, dH^{\#}) = \sum_{r=1}^{k+1} \max_{s=r,M} |D^s(F-H)|, \qquad \text{for } k < \infty$$

and

$$d_\infty(dF^{\#}, dH^{\#}) = \sum_{r=1}^{\infty} \frac{2^{-r}d_r}{1+d_r}, \qquad \text{for } k = \infty.$$

(Here we take advantage of the fact that: given $\epsilon_0(x) > 0$, then $|H(Q_0)|$ and some estimate $|DH(x)| < \epsilon_1(x)$, imply that $|H(x)| < \epsilon_0(x)$ on M).

Whether or not M is compact, open-dense subsets of $\mathfrak{C}^{k+1}$ project onto open-dense subsets of $\mathfrak{H}^k$. In fact, generic subsets of $\mathfrak{C}^{k+1}$ project onto generic subsets of $\mathfrak{H}^k$, as defined next.

Definition. Let M be a symplectic manifold with Hamiltonian systems $\mathfrak{H}^k$ for fixed $k = 1, 2, \ldots, \infty$. A subset $\mathfrak{P} \subset \mathfrak{H}^k$, or the defining property logically specifying the subset $\mathfrak{P}$, is generic in case $\mathfrak{P}$ contains a residual subset of $\mathfrak{H}^k$.

Note that, while a set of Hamiltonian systems may be generic, no individual system is generic—although we sometimes make such an assertion in an informal discussion, e.g. see the title of this paper. As an example, we next proceed to define a generic subset $\mathfrak{S}_0$ of $\mathfrak{H}^k$ by requiring generic behavior at each critical point.

Definition. A critical point Q_0 of a Hamiltonian system $dH^\# \in \mathfrak{H}^k$ on the symplectic manifold M is generic in case: the eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n, -\lambda_1, -\lambda_2, \ldots, -\lambda_n)$ at Q_0 are distinct and $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ are linearly independent over the rational number field. (Note: the replacement of λ_j by $-\lambda_j$ does not affect the condition.)

THEOREM 1. *Let $\mathfrak{H}^k$, for fixed $k = 1, 2, \ldots, \infty$, be the Hamiltonian systems on a symplectic manifold M. Then the subset $\mathfrak{S}_0{}^k$ is generic in $\mathfrak{H}^k$, where we define:*

$$\mathfrak{S}_0{}^k = \{dH^\# \in \mathfrak{H}^k \mid \textit{every critical point of } dH^\# \textit{ is generic}\}.$$

Proof. By the techniques developed in Theorem 2 of the Memoir [9], the space $\mathfrak{C}^{k+1}$ of Hamiltonian C^{k+1}-functions on M has a dense subset consisting of C^∞-Hamiltonians having only generic critical points. Using the projection $\pi: \mathfrak{C}^{k+1} \to \mathfrak{H}^k$ we obtain a dense set $\hat{\mathfrak{H}} \subset \mathfrak{H}^k$ consisting of C^∞-Hamiltonian systems having only generic critical points.

Fix an integer $N \geq 2$ and call a critical point of $H \in \mathfrak{H}^k$ N-generic in case: the eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n, -\lambda_1, -\lambda_2, \ldots, -\lambda_n)$ of H there are distinct and satisfy the condition, $\Sigma^n{}_{j=1}\, \alpha_j \lambda_j \neq 0$ for all sets of integers $\{\alpha_j\}$ with $0 < \Sigma|\alpha_j| \leq N$. Then, since perturbations within the Whitney topology can be localized on M, there exists an open neighborhood $\mathfrak{H}^k(N)$ of the set $\hat{\mathfrak{H}}$ in $\mathfrak{H}^k$ such that each Hamiltonian system in $\mathfrak{H}^k(N)$ has only N-generic critical points. Clearly $\mathfrak{S}_0{}^k$ contains the residual set $\bigcap_{N\geq 2} \mathfrak{H}^k(N)$, and so $\mathfrak{S}_0{}^k$ is generic in $\mathfrak{H}^k$. □

In the next section we shall introduce other generic subsets $\mathfrak{S}_1{}^k$, $\mathfrak{S}_2{}^k$, $\mathfrak{S}_3{}^k$ of $\mathfrak{H}^k$, as specified by conditions imposed on their periodic orbits. For example Robinson [14] has defined a generic subset $\mathfrak{R}^k \subset \mathfrak{H}^k$, for each $k \geq 2$, consisting of Hamiltonian systems almost all of whose periodic orbits are nondegenerate (that is, all excepting a possible countable number of degenerate periodic orbits) with distinct characteristic multipliers.

We shall generalize this result, following a procedure of Takens [16], by imposing generic conditions on the Poincaré maps around the periodic orbits of Hamiltonian dynamical systems. Recall that for any periodic orbit γ of a Hamiltonian system $dH^{\#} \in \mathfrak{H}^k$, on a symplectic $2n$-manifold M, the behavior of the solutions in a tubular neighborhood about γ can be analyzed by means of the Poincaré section map P of a transversal $(2n - 1)$-section Σ into itself, upon following the trajectories for $dH^{\#}$ for a single encirclement of the tube. It is always possible to introduce local canonical coordinates $(x^1, x^2, \ldots, x^n, y_1, y_2, \ldots, y_n)$ with origin at $\Sigma \cap \gamma$ so that $H = y_1$ and also Σ is specified by $x^1 = 0$, see [1]. Then appropriate local coordinates on Σ are given by $y_1 = h$ (the energy level) and $(x^2, \ldots, x^n, y_2, \ldots, y_n)$. Thus Σ can be parametrized by h as a union of slices $\Sigma(h)$ each of which is an open set in the linear symplectic space $\mathbf{R}^{n-1} \oplus \mathbf{R}^{n-1}$. Takens defines the parameter-symplectic space to be the product $\mathbf{R} \times (\mathbf{R}^{n-1} \oplus \mathbf{R}^{n-1})$, bearing the canonical 2-form $\Sigma^n_{j=2}\, dx^j \wedge dy_j$, and we recognize the section Σ as a neighborhood of the origin in this linear space (often abbreviated as $\mathbf{R}^{2n-1}$).

Then the Poincaré map P maps an open neighborhood W of the origin in $\mathbf{R} \times (\mathbf{R}^{n-1} \oplus \mathbf{R}^{n-1})$ into this same parameter-symplectic space by

$$y_1 \to y_1 = h,$$

and then a symplectic map on each energy level h in W,

$$P_h : (x^j, y_j) \to (X^j, Y_j) \quad \text{for} \quad j = 2, \ldots, n.$$

In the terminology of Takens, which is clarified below, P is a parameter-symplectic map of W, and the geometry of H near γ in M determines P, at least up to a parameter-symplectic automorphism of $W \subset \mathbf{R} \times (\mathbf{R}^{n-1} \oplus \mathbf{R}^{n-1})$.

The map P and its l-jet extension $P^{(l)}$ (the l-th order truncation of the Taylor series of P about each point in W, as indicated below), are thus geometric invariants of the Hamiltonian system $dH^{\#}$ near γ. For instance $P^{(1)}$ at γ is given by the Jacobian matrix

$$dP_0 = \left(\frac{\partial(X, Y)}{\partial(x, y)} \right)_0 \in Sp(2n - 2, \mathbf{R})$$

with eigenvalues $(\mu_2, \ldots, \mu_n, \mu_2^{-1}, \ldots, \mu_n^{-1})$, together with the vector

$$\begin{pmatrix} \partial X/\partial h \\ \partial Y/\partial h \end{pmatrix}_0 .$$

Of course, each parameter-symplectic automorphism of $\mathbf{R} \times (\mathbf{R}^{n-1} \oplus \mathbf{R}^{n-1})$, with origin fixed, induces an inner automorphism of $Sp(2n-2, \mathbf{R})$ and replaces dP_0 by a similar symplectic matrix.

We next present a brief exposition and summary of Takens' approach, leading to his Theorem A [16] reformulated in our proposition below.

Parameter-symplectic jet spaces $J^l(2n-1)$. The product $\mathbf{R} \times (\mathbf{R}^{n-1} \oplus \mathbf{R}^{n-1})$, with the 2-form $\Sigma^n_{j=2}\, dx^j \wedge dy_j$ in the coordinates $(h, x^2, \ldots, x^n, y_2, \ldots, y_n)$, is called the standard parameter-symplectic space of dimension $2n-1$. A diffeomorphism between open subsets of $\mathbf{R} \times (\mathbf{R}^{n-1} \oplus \mathbf{R}^{n-1})$ which maps constant energy levels ($h =$ const.) into the same energy levels, and which preserves the given symplectic 2-form on each such energy level, is called a parameter-symplectic map. In the case of the transversal section Σ to the periodic orbit γ of $dH^{\#}$, we note that any two parameter-symplectic charts on Σ are related by a parameter-symplectic map. It is in this sense that the Poincaré map P on $\Sigma \subset \mathbf{R} \times (\mathbf{R}^{n-1} \oplus \mathbf{R}^{n-1})$ is a parameter-symplectic map, determined to within a parameter-symplectic conjugation.

Let $J^0 = J^0(W, \mathbf{R}^{2n-1})$ be the space of all zero-jets of parameter-symplectic maps of $W \subset \mathbf{R}^{2n-1}$ into $\mathbf{R}^{2n-1}$, without any special reference to the origin. That is, take point pairs

$$J^0 = \{(p_1, p_2) \mid p_1 \in W,\ p_2 \in \mathbf{R}^{2n-1} \text{ and } y_1(p_1) = y_1(p_2)\}.$$

Naturally the topology and manifold structure of J^0 are defined by considering J^0 as a subset of the product $W \times \mathbf{R}^{2n-1}$ (and similar statements hold for the higher order jet spaces). An important subset of J^0 is the manifold called FIX defined by

$$\text{FIX} = \{(p_1, p_2) \mid p_1 \in W,\ p_2 \in \mathbf{R}^{2n-1} \text{ and } p_1 = p_2\}.$$

The codimension of FIX in J^0 is just

$$\dim J^0 - \dim \text{FIX} = [(2n-1) + (2n-1) - 1] - [(2n-1)] = 2n-2.$$

For the Poincaré map P around the periodic orbit γ, the zero-jet at each point $(h, z) \in W$ is $(h, z; h, Z)$ where $z = (x^2, \ldots, x^n, y_2, \ldots, y_n)$ and $Z = (X^2, \ldots, X^n, Y_2, \ldots, Y_n)$. The zero-jet extension is the map

$$P^{(0)}: W \to J^0: (h, z) \to (h, z; h, Z).$$

Note that $P^{(0)}$ is nonsingular on W and so the image of W is a $(2n - 1)$-manifold in J^0. If $P^{(0)}$ is transversal to FIX $\subset J^0$, then the inverse image of FIX, namely $(P^{(0)})^{-1}(\text{FIX})$, has components in W that are 1-dimensional submanifolds. This follows from general transversality theory (see [7 p. 23, 44 and 17 p. 270] and the clarifying comments below), and the facts concerning dimensions:

$$\operatorname{codim} \text{FIX} = 2n - 2 \quad \text{and} \quad \dim W = 2n - 1.$$

Of course, the set $(P^{(0)})^{-1}(\text{FIX})$ describes the fixed points of P in W and hence it designates the periodic orbits of $dH^{\#}$ meeting W and encircling the tubular neighborhood of γ just once.

Next consider the l-jet extension of P,

$$P^{(l)}: W \to J^0 \times J^l(2n - 1), \text{ each } 1 \leq l < k,$$

where $J^l(2n - 1)$ is the Lie group of l-jets of parameter-symplectic maps $\alpha: (\mathbf{R}^{2n-1}, 0) \to (\mathbf{R}^{2n-1}, 0)$ with the origin fixed.

For instance $J^1(2n - 1) = Sp(2n - 2, \mathbf{R}) \times \mathbf{R}^{2n-2}$, and $J^l(2n - 1)$ is also a real analytic manifold, see lemma below. Furthermore, just as for $P^{(0)}$, the map $P^{(l)}$ is a nonsingular C^{k-l}-map of W, and so defines a topological embedding of W as a C^{k-l}-submanifold $P^{(l)}(W)$ in the real analytic manifold $J^0 \times J^l(2n - 1)$.

For any prescribed suitable closed subset (e.g. submanifold) $Q \subset J^l(2n - 1)$ we look for the transversality of $P^{(l)}(W)$ with $(\text{FIX}) \times Q \subset J^0 \times J^l(2n - 1)$, a condition we abbreviate as: P is Q-transversal. If P is Q-transversal, then the subset $(P^{(l)})^{-1}((\text{FIX}) \times Q)$ in W corresponds to the periodic orbits of $dH^{\#}$ whose Poincaré maps satisfy the l-jet condition specified by Q.

Q-transversality. In order to make precise the conceptual basis for our applications of transversality theory, we consider the space $\mathfrak{C}^k(W, \mathbf{R}^{2n-1})$ of all C^k-maps, for fixed $k = 1, 2, \ldots, \infty$, of the open set $W \subset \mathbf{R}^{2n-1}$ into this same real number space. We use the Whitney C^k-topology on $\mathfrak{C}^k(W, \mathbf{R}^{2n-1})$, but mention that this reduces to the metric uniform C^k-topology if we should replace W by some compact subset $W_c \subset W$.

Next let Q be a given closed subset of $J^l(2n - 1)$, for some chosen positive integer $l < k$. Usually Q is a topologically embedded C^∞-submanifold, although we can allow Q to a real analytic variety, or even a semi-analytic set or any other W-object satisfying the axioms of Whitney [17 p. 264, 271 and 19, 20]. We assume furthermore that

$$\operatorname{codim} Q = \dim J^l(2n - 1) - \dim Q \geq 1.$$

Take a map $\varphi \in \mathfrak{C}^k(W, \mathbf{R}^{2n-1})$ and consider the l-jet extension

$$\varphi^{(l)}: W \to J^0 \times J^l(2n - 1).$$

We define φ to be Q-transversal on W in case the image $\varphi^{(l)}(W)$ is transversal to the given set (FIX) $\times$ $Q \subset J^0 \times J^l(2n - 1)$ at every point of intersection. In this case the inverse image $(\varphi^{(l)})^{-1}((\text{FIX}) \times Q)$ consists of countably many components, each isolated in W within an open neighborhood meeting no other components. Furthermore each such component is a topologically embedded C^1-submanifold, and only a finite number of these meet any prescribed compact subset $W_c \subset W$.

If we require $\varphi \in \mathfrak{C}^k(W, \mathbf{R}^{2n-1})$ to be Q-transversal only on the compact set W_c, then classical transversality theory [17 p. 270, 271] asserts: *there exists an open and dense set of maps in* $\mathfrak{C}^k(W, \mathbf{R}^{2n-1})$ *each of which is Q-transversal on W_c.*

In our geometric analysis of Hamiltonian systems $dH^\# \in \mathfrak{H}^k$ on M, the set $Q \subset J^l(2n - 1)$ must be specified as an intrinsic geometric locus without regard to the particular parametric-symplectic coordinates on the transverse section W to the periodic orbit γ. This means that Q must be invariant under every inner automorphism of the group $J^l(2n - 1)$.

We combine these two types of demands on Q, namely regularity and invariance, under the definition of a normal set. Thus a closed set $Q \subset J^l(2n - 1)$ will be called a *normal set* in case:

i) Q is a C^∞-submanifold topologically embedded as a closed set in $J^l(2n - 1)$; or else Q is a real analytic variety (zeros of some real analytic function), or a closed semi-analytic set (locally defined by a finite number of conditions $f_i = 0$, $f_j \geq 0$ for real analytic functions f_i and f_j, as discussed later), and

ii) $\alpha^{-1}Q\alpha = Q$, as a set, for each group element $\alpha \in J^l(2n - 1)$.

If Q is a normal set in $J^l(2n - 1)$, then we can define the periodic orbit γ of $dH^\# \in \mathfrak{H}^k$ to be Q-transversal in case there exists a transverse section W to γ in M for which the Poincaré map P is Q-transversal.

In these terms we now state a proposition, a slightly modified version of Takens' Theorem A [16], concerning the space $\mathfrak{H}^k$, for fixed $1 \leq k \leq \infty$, of Hamiltonian vector fields on a symplectic manifold M of dimension $2n \geq 4$.

PROPOSITION. *Let Q be a normal subset of $J^l(2n - 1)$. Then there exists a generic set $\tau_Q{}^k \subset \mathfrak{H}^k$ (for each fixed $1 \leq l \leq k \leq \infty$) such that: each periodic orbit γ of every $dH^\# \in \tau_Q{}^k$ is Q-transversal.*

Remark 1. The periodic orbit γ of $dH^\# \in \mathfrak{H}^k$ is Q-transversal just in case the corresponding Poincaré map P of some transverse section W to γ is Q-transversal. This is the terminology used by Takens, but he phrases his Theorem A with reference to the Baire space $\mathfrak{C}^{k+1}$ of Hamiltonian functions on M, and with reference to the differentiability classes $l + 1 = k < \infty$ (although his proof also allows any finite l and $k = \infty$, as asserted in private discussions).

The usage of $\mathfrak{H}^k$ in place of $\mathfrak{C}^{k+1}$ is possible since Takens utilizes only subsets of $\mathfrak{C}^{k+1}$ that are specified by demands on the corresponding Hamiltonian vector fields. But generic subsets of $\mathfrak{C}^{k+1}$, which are unions of normalized Hamiltonians, determine generic subsets of $\mathfrak{H}^k$, in accord with our earlier topological considerations.

The differentiability restrictions can also be relaxed to $l < k \leq \infty$ by an easy argument. If Q is normal in $J^l(2n - 1)$, then it is also normal as a subset of $J^{l'}(2n - 1)$ for $l' = k - 1 < \infty$ (referring to the standard embedding of $J^l(2n - 1)$ in $J^{l'}(2n - 1)$). Then there exists a generic class $\hat{\tau}_Q{}^k \subset \mathfrak{H}^k$ for which each Poincaré map P has an l'-jet extension that is Q-transversal, and so the l-jet extension $P^{(l)}$ is transversal to (FIX) $\times\, Q \subset J^0 \times J^l(2n - 1)$, on some section W as required.

Remark 2. Usually Q is prescribed as a normal set in some $J^l(2n - 1)$ for given positive integer l, and then attention is focused on $\mathfrak{H}^k$ for appropriate $l + 1 \leq k \leq \infty$ in order to find a generic set $\tau_Q{}^k \subset \mathfrak{H}^k$. Since the particular value of k is often not important, we use the notation τ_Q for the required generic set in any suitable $\mathfrak{H}^k$ of interest. By Theorem 1 above, we can further assume that each Hamiltonian system $dH^\# \in \tau_Q$ has only isolated critical points, and henceforth we shall impose this demand whenever convenient.

Remark 3. Usually the normal set Q will be specified by certain polynomial identities involving the components of $J^l(2n-1)$. That is, we consider $J^l(2n-1)$ to be a subset of the real linear space $M_0{}^l(2n-1)$ consisting of all l-jets of maps of the $(2n-1)$-vector space into itself, while holding the origin fixed. In the lemma below we prove that $J^l(2n-1)$ is a real analytic manifold, which is a closed subset in the linear space $M_0{}^l(2n-1)$. Then we can consider Q as an analytic variety in $J^l(2n-1)$, with the same local properties as an analytic variety in some real linear space. That is, locally Q has the form $f = 0$ for some real analytic function f on $J^l(2n-1)$.

In more detail let $\hat{Q}$ be a real polynomial in the cartesian coordinates of $M_0{}^l(2n-1)$, with a corresponding zero-set $[\hat{Q}]$ that constitutes a real algebraic variety in the number space $M_0{}^l(2n-1)$. Whitney [19] has proved that $[\hat{Q}]$ consists of a finite disjoint collection of real analytic submanifolds of $M_0{}^l(2n-1)$, each topologically embedded. Since $J^l(2n-1)$ is also a real analytic manifold, the intersection $Q = [\hat{Q}] \cap J^l(2n-1)$ is a real analytic variety. Thus Q is a (Whitney) W-object, whether regarded as a closed subset of $M_0{}^l(2n-1)$ or in $J^l(2n-1)$, and is an appropriate target set in general transversality theory. Such a set $Q \subset J^l(2n-1)$ is normal provided it remains invariant under all inner automorphisms of the Lie group $J^l(2n-1)$.

LEMMA. *The parameter-symplectic jet space $J^l(2n-1)$ is a closed subgroup of the Lie group $L^l(2n-1)$ of all invertible l-jets on $(\mathbf{R}^{2n-1}, 0)$. Thus each component of $J^l(2n-1)$ is a real analytic submanifold, topologically embedded as a closed set in $M_0{}^l(2n-1)$.*

Proof. Let $\mathfrak{C}_0{}^l(2n-1)$, for fixed positive integer l, denote the group of germs of invertible C^l-maps of the real $(2n-1)$-number space into itself with the origin fixed. Then $L^l(2n-1)$ is the group of l-jets of the maps of $\mathfrak{C}_0{}^l(2n-1)$, and so $L^l(2n-1)$ is an open subset of $M_0{}^l(2n-1)$. In fact, $L^l(2n-1)$ is a Lie group analytically diffeomorphic with the manifold $GL(2n-1, \mathbf{R}) \times \mathbf{R}^\sigma$, where

$$\sigma = \sum_{r=1}^{l} \binom{2n-2+r}{r} \cdot (2n-1)$$

is the appropriate number of independent coefficients among the nonlinear terms, see [7 p. 6].

Clearly $J^l(2n - 1)$ is a subgroup of the Lie group $L^l(2n - 1)$; moreover, it lies within the closed set corresponding to $SL(2n - 1, \mathbf{R}) \times \mathbf{R}^\sigma$. When we show that $J^l(2n - 1)$ is a closed subgroup of $L^l(2n - 1)$, then the conclusions of the lemma will follow by general Lie theory. For this purpose consider any sequence $f_1, f_2, \ldots, f_k, \ldots$ of germs of parameter-symplectic maps in $\mathfrak{C}_0{}^l(2n - 1)$, say of the form

$$(h, z) \rightarrow (h, f(h, z))$$

in terms of parametric-symplectic coordinates $(h, z) = (h, x, y)$ on $\mathbf{R} \times (\mathbf{R}^{n-1} \oplus \mathbf{R}^{n-1})$. Let the corresponding l-jets be denoted by $(1, f_k{}^l) \in J^l(2n - 1)$ and we assume convergence, $\lim_{k\to\infty} f_k{}^l = f_\infty{}^l$ within $L^l(2n - 1)$. We wish to construct a parameter-symplectic germ $(h, f_\infty(h, z))$ yielding the required l-jet $f_\infty{}^l$. Using the group properties of $J^l(2n - 1)$ we can assume that the limiting jet $f_\infty{}^l$ is very near the identity of $L^l(2n - 1)$. Also, by composing maps with a translation $z \rightarrow z + t(h)$ on each slice $h =$ constant, we can assume that the jet $f_\infty{}^l$ corresponds to some map of the form $(h, z) \rightarrow (h, f_\infty(h, z))$ with $f_\infty(h, z) = B(h)z + \cdots$. In fact, after modifying the members of the approximating sequence, we can assume that each $f_k(h, z) = B_k(h)z + \cdots$.

The basic symplectic condition, $T'JT = J$ or equally well $TJT' = J$ in terms of the Jacobian matrix $T = \partial f/\partial z$, is nonlinear and moreover it mixes the components of the l-jet f^l into confusing nonhomogeneous identities. For this reason it is difficult to extend a given l-jet such as $f_\infty{}^l$ to a parameter-symplectic map on a neighborhood of the origin. To bypass these difficulties we "linearize the problem" by means of a generating function that transforms germs of Hamiltonian vector fields into germs of symplectic maps.

The motivation for our choice of generating function is the study of the matricial linear fractional transformation

$$B = (I + A)(I - A)^{-1},$$

with the inverse relation (noting $AB = BA$),

$$A = (B - I)(B + I)^{-1}.$$

Here each square matrix A (with no eigenvalue $+1$) is transformed to a matrix B (with no eigenvalue -1), and vice versa. Moreover A is Hamiltonian if and only if B is symplectic. For instance, assume B symplectic so $B'J = JB^{-1}$, $(B')^2J = JB^{-2}$, etc. and compute

$$A'J = (B' + I)^{-1}(B' - I)J = (B' + I)^{-1}J(B^{-1} - I)$$

$$= J(B^{-1} + I)^{-1}(B^{-1} - I)$$

$$= J[B(B^{-1} + I)]^{-1}B(B^{-1} - I) = J(I + B)^{-1}(I - B) = -JA.$$

Similarly, $A'J + JA = 0$ implies that $B'JB = J$, see [10] for this calculation and further discussions of the generating function.

Now we generalize this formula to nonlinear vector fields and maps. In the space $\mathbf{R}^{2n-1}$, in which we fix the coordinates (h, z) as above, consider any germ of a "horizontal" C^l-vector field with origin as critical point, say having the components $(0, g)$ where

$$g(h, z) = A(h)z + \cdots.$$

We use this vector field to generate the germ of a C^l-diffeomorphism $(h, z) \to (h, f(h, z))$, where

$$f(z) = (i + g) \circ (i - g)^{-1}(z).$$

Here i denotes the identity map on $\mathbf{R}^{2n-2}$, and we suppress any explicit mention of the parameter h for the time being. In order to clarify this functional equation define $\chi(z)$ implicitly by $(i - g)^{-1}(z) = \chi(z)$ or $z = (i - g)\chi = \chi - g(\chi)$. Then we compute

$$f(z) = (i + g)\chi(z) = \chi(z) + g(\chi(z)) = 2\chi(z) - z$$

and we note that $f(z) = B(h)z + \cdots$ where

$$B = (I + A)(I - A)^{-1}.$$

Furthermore, it is not difficult to show that each such germ of a map $f(h, z) = B(h)z + \cdots$ determines $g(h, z)$ by the inverse relation

$$g(z) = \frac{1}{2}(f - i) \circ (f + i)^{-1}(2z),$$

provided the matrices A and B have no eigenvalues $+1$ and -1, respectively.

The chain rule of differentiation replaces functional composition by the multiplication of Jacobian matrices (now written $\partial g = \partial g / \partial z$) and so we compute

$$\partial f(z) = (I + \partial g)(I - \partial g)^{-1}(z),$$

for all z near the origin. Thus ∂g is everywhere Hamiltonian if and only if ∂f is everywhere symplectic, near $z = 0$.

Next let us consider the l-jets g^l and f^l. By the usual calculus of power series, the components of f^l are each real analytic functions (in fact, rational functions over the real rationals) of the components of g^l, and vice versa. Thus the transformations $g^l \to f^l$ and $f^l \to g^l$ are continuous.

Finally assume that $g(h, z) = J(\partial H / \partial z)$ is a Hamiltonian vector field for some given Hamiltonian function $H(h, z)$, while treating h as a parameter. Then $\partial g = J(\partial^2 H / \partial z^2)$ is a Hamiltonian matrix at each point near the origin, and conversely $g = J(\partial H / \partial z)$ holds provided

$$(\partial g)' J + J(\partial g) = 0.$$

In such a case ∂f is everywhere symplectic, and so $(h, f(h, z))$ is a parameter-symplectic map on $\mathbf{R}^{2n-1}$.

Moreover the linear Hamiltonian condition, ignoring h, on ∂g imposes a finite number of linear constraints on the components of the corresponding l-jet g^l. Moreover, each of these linear constraints involves only a single order of the partial derivatives of g at $z = 0$. Furthermore any such assignment of data compatible with these finite number of linear constraints does yield a jet g^l that can be realized by a global Hamiltonian that is merely a polynomial of degree $(l + 1)$. In this way we conclude that the set of all components of l-jets for Hamiltonian vector fields $g(z) = Az + \cdots$ on $\mathbf{R}^{n-1} \oplus \mathbf{R}^{n-1}$, constitutes a closed set in the corresponding number space.

Because of the continuity of the transformations $g^l \leftrightarrow f^l$, we conclude that the l-jet space $J_s{}^l(2n - 2)$ of symplectomorphism germs is also a closed set in $L^l(2n - 2)$. The same argument, applied to the derivatives of ∂g with respect to h, shows that $J^l(2n - 1)$ is a closed subset of $L^l(2n - 1)$, as required. □

Remark. Since $J^l(2n - 1)$ is a closed Lie subgroup of $L^l(2n - 1)$ only finitely many components can meet any prescribed compact subset of $M_0{}^l(2n - 1)$. It seems likely that $J^l(2n - 1)$ is even a connected algebraic variety, as is the case of $J^1(2n - 1) \approx Sp(2n - 1) \times \mathbf{R}^{2n-2}$.

We now are able to interpret the above general proposition concerning periodic orbits of generic Hamiltonian systems, in terms of two corollaries that will be immediately applicable to our subsequent theory. The first corollary deals with topological conditions, and the second with algebraic conditions relevant in transversality analysis.

COROLLARY 1. *Let Q be normal in $J^l(2n - 1)$ with corresponding generic set $\tau_Q \subset \mathfrak{H}^k$ for $1 \leq l < k \leq \infty$ as in the above proposition.*

Let codim $Q = 1$. *Then, for every $dH^\# \in \tau_Q$, each periodic orbit γ with Poincaré l-jet extension $P^{(l)}$ on W, is isolated in the sense:*

$$\Delta = (P^{(l)})^{-1}((\mathrm{FIX}) \times Q)$$

consists of isolated points in some transversal W. Thus for each $dH^\# \in \tau_Q$ there are only a countable number of "exceptional periodic orbits" in M, with Poincaré l-jet satisfying the condition Q.

Let codim $Q > 1$. *Then, for each $dH^\# \in \tau_Q$ no periodic orbit in M has a Poincaré l-jet with the condition Q.*

Proof. Consider a transversal section W through a periodic orbit γ of $dH^\# \in \tau_Q$. We have an easy computation

$$\dim W + \dim((\mathrm{FIX}) \times Q) = \dim(J^0 \times J^l(2n - 1)) + \dim \Delta.$$

Take codim $Q = \dim J^l(2n - 1) - \dim Q$, and note that

$$\dim \Delta = \dim W - [\dim J^0 - \dim(\mathrm{FIX})] - \operatorname{codim} Q = 1 - \operatorname{codim} Q.$$

If codim $Q = 1$, then each component of Δ is a 0-dimensional manifold, and so Δ consists of isolated points in a suitably small W.

Let V be any compact subset of M whereon $dH^\#$ is noncritical, and let $L = [1/N, N]$ be a compact time duration, as specified by a positive integer N. Define $S(V, L)$ to be the set of all periodic orbits of $dH^{\#}$, each of which meets the set V, has least period in L, and which has the property Q for its corresponding Poincaré return map.

Suppose the set of orbits $S(V, L)$ were un-countable. Then there would exist an accumulation orbit $\tilde{\gamma}$ of $dH^{\#}$, such that every tubular neighborhood of $\tilde{\gamma}$ contains an un-countable number of periodic orbits of $S(V, L)$. Of course the periodic orbit $\tilde{\gamma}$ meets V, and its least period of return to a transversal $\tilde{W}_1$ is some $\tau \leq N$; but nothing is said concerning its Poincaré map $\tilde{P}$. Restrict $\tilde{W}_1$ so that the time of first return for any of its points is greater than $\tau/2$, and take a positive integer k such that $k \cdot (\tau/2) \geq N$. Then choose a still smaller transversal for $\tilde{\gamma}$, namely $\tilde{W}_2 \subset \tilde{W}_1$ so that $\tilde{P}$ and its first k iterates $\tilde{P}^2, \tilde{P}^3, \ldots, \tilde{P}^k$ all map $\tilde{W}_2$ into $\tilde{W}_1$.

With this geometric situation we see that each orbit γ^* of $S(V, L)$ meeting $\tilde{W}_2$, must have $\tilde{W}_2$ as a transversal section and must have a Poincaré first-return map of either $\tilde{P}$, or else one of the iterates $\tilde{P}^2, \ldots, \tilde{P}^k$. Thus we can classify the orbits of $S(V, L)$ meeting $\tilde{W}_2$ into k disjoint classes, depending on the iterate of $\tilde{P}$ corresponding to the first-return to $\tilde{W}_2$.

By the first part of the above proof the periodic orbits of $dH^{\#} \in \tau_Q$, having property Q for the Poincaré map $\tilde{P}$ on a suitably small transversal section in $\tilde{W}_2$, must describe a set of isolated points in $\tilde{W}_2$, and hence constitute a finite or denumerable infinite set, at most. Similar assertions hold for those orbits of $dH^{\#}$ corresponding to the Poincaré map $\tilde{P}^2$ on $\tilde{W}_2$, and for each of the classes corresponding to $\tilde{P}^3, \ldots, \tilde{P}^k$. Hence only a countable number of orbits of $S(V, L)$ meet $\tilde{W}_2$; but this contradicts the choice of the accumulation orbit $\tilde{\gamma}$. Thus we conclude that $S(V, L)$ is countable (finite or denumerable infinity, at most).

The countable set of "exceptional periodic orbits" of $dH^{\#}$ (those whose Poincaré l-jet meets Q) is merely the union of all $S(V, L)$, as V and L run through denumerable sequences of compact subsets exhausting the manifold (M-critical points of $dH^{\#}$) and the half-line $(0, \infty)$, respectively.

Finally assume codim $Q > 1$. Then $\dim \Delta \leq -1$ and in this case Δ must be empty. □

COROLLARY 2. *Let Q_1 and Q_2 be normal subsets of $J^l(2n-1)$, each a non-empty proper subset defined by the zero-set of a real polynomial, as above. Then* codim $Q_1 \geq 1$.

Further assume that the difference set $Q_1 - Q_2$ *is dense in* Q_1. *Then* $Q_1 \cap Q_2$ *is normal in* $J^l(2n-1)$, *and* codim $(Q_1 \cap Q_2) \geq 2$.

Proof. By the geometric analysis of Whitney [19, 20] the W-object Q_1 is the intersection of a finite disjoint collection of analytic submanifolds (each connected but not necessarily closed) in $J^l(2n-1)$. Since $Q_1 \neq J^l(2n-1)$, the analyticity property requires that each of these submanifolds has codimension ≥ 1. Hence codim $Q_1 \geq 1$ in $J^l(2n-1)$.

Next consider the normal set $Q_1 \cap Q_2$ in $J^l(2n-1)$. Suppose Q_1 and Q_2 each contains a submanifold of codimension 1, and further suppose that these two hypersurfaces intersect in a nonempty piece of hypersurface in $J^l(2n-1)$. But in such a case $Q_1 - Q_2$ is not dense in Q_1, contradicting the hypothesis. Hence we conclude that no such hypersurface lies in $Q_1 \cap Q_2$, so codim $(Q_1 \cap Q_2) \geq 2$. □

Remarks. 1) These intersection and density criteria are especially easy to apply in $J^1(2n-1) = Sp(2n-2, \mathbf{R}) \times \mathbf{R}^{2n-2}$ wherein we can usually diagonalize the matrices in $Sp(2n-2, \mathbf{R})$.

2) Also the conclusion of Corollary 2, namely codim $Q \geq 1$, with the resulting applications of Corollary 1, is valid in the case where Q is a real analytic variety. For instance we could take Q as in the intersection of $J^l(2n-1)$ with the zero-set $[\hat{Q}]$ of a real analytic function $\hat{Q}$ in $M_o{}^l(2n-1)$.

A further technical extension of these ideas allows Q to be any closed subset of $J^l(2n-1)$ defined by $[\hat{Q}] \cap A^l$, where A^l is a closed semi-analytic set in $J^l(2n-1)$ and $\hat{Q}$ is a real analytic function in some open neighborhood N^l of A^l, see [17 p. 270, 271 and 20]. The most general case we shall encounter assumes that A^l is defined in $J^l(2n-1)$ by the locus $f \geq 0$ where f is a real analytic function on the open set $N^l \subset J^l(2n-1)$ and $f < 0$ near the boundary of N^l.

In the application of these corollaries and remarks, we shall always assume that Q is a nonempty proper subset of $J^l(2n-1)$, and that Q is invariant under all inner automorphisms of $J^l(2n-1)$. We now apply these ideas to the concept of a generic periodic orbit, as defined in terms of the corresponding Poincaré map and the related characteristic polynomial.

Each symplectic matrix $T \in Sp(2n-2, \mathbf{R})$ has a characteristic polynomial

$$\det|\mu I - T| = \mu^{2n-2} + c_{2n-3}\mu^{2n-3} + c_{2n-4}\mu^{2n-4} + \cdots + c_1\mu + c_0.$$

where the characteristic coefficients $c_0, c_1, \ldots, c_{2n-3}$ are known to be elementary symmetric polynomials in the eigenvalues $\mu_2, \mu_3, \ldots, \mu_n, \mu_2^{-1}, \mu_2^{-1}, \ldots, \mu_n^{-1}$. For instance, $\det T = \mu_2\mu_3 \cdots \mu_n\mu_2^{-1} \cdots \mu_n^{-1} = c_0$, and $\text{Trace } T = \mu_2 + \mu_3 + \cdots + \mu_n + \mu_2^{-1} + \mu_3^{-1} + \cdots + \mu_n^{-1} = -c_{2n-3}$. Note that $c_0 = 1$, or that the polynomial $X_0 - 1$ is annihilated by every set of characteristic coefficients for every symplectic matrix, and thus this polynomial $X_0 - 1$ is useless for making distinctions among the matrices of $Sp(2n - 2, \mathbf{R})$.

Definition. A symplectic polynomial $Q(X_0, X_1, \ldots, X_{2n-3})$ is a polynomial in the $(2n - 2)$ indeterminants $(X_0, X_1, \ldots, X_{2n-3})$, with rational coefficients, such that $Q(c_0, c_1, \ldots, c_{2n-3}) \neq 0$ as evaluated on the characteristic coefficients of some matrix $T \in Sp(2n - 2, \mathbf{R})$.

Of course each symplectic polynomial can also be interpreted as a polynomial in the components of $J^1(2n - 1) \subset M_0{}^1(2n - 1)$, and thus its zero-set defines an algebraic variety that is a normal subset of $J^1(2n - 1)$.

Definition. A periodic orbit γ of a Hamiltonian system $dH^\# \in \mathfrak{H}^k$ on a symplectic manifold M is generic in case:

The characteristic coefficients $(c_0, c_1, \ldots, c_{2n-3})$ of the corresponding Poincaré matrix dP_0 satisfy no symplectic polynomial.

THEOREM 2. *Let $\mathfrak{H}^k$, for fixed $k = 2, 3, \ldots, \infty$, be the Hamiltonian systems on a symplectic manifold M. Then the subset $\mathfrak{R}_0$ is generic in $\mathfrak{H}^k$, where we define*

$$\mathfrak{R}_0 = \{dH^\# \in \mathfrak{H}^k \mid \textit{all but a countable number of the periodic orbits of } dH^\# \textit{ are generic}\}.$$

Proof. Take one symplectic polynomial $Q_1(X_0, X_1, \ldots, X_{2n-3})$ which thereby defines an algebraic subvariety Q_1 of $J^1(2n - 1) = Sp(2n - 2, \mathbf{R}) \times \mathbf{R}^{2n-2}$. Since Q_1 is specified by the matrices $T \in Sp(2n - 2, \mathbf{R})$ whose characteristic coefficients annihilate $Q_1(X_0, X_1, \ldots, X_{2n-3})$ we note that Q_1 is a normal subset of $J^1(2n - 1)$.

By the Corollary 2 above, codim $Q_1 \geq 1$. Hence by Corollary 1, there is a generic set $\tau_{Q_1} \subset \mathfrak{H}^k$ such that: for each $dH^\# \in \tau_{Q_1}$ there are only a countable number of "exceptional periodic orbits" in M having Poincaré matrix dP_0 whose characteristic coefficients $c_0, c_1, \ldots, c_{2n-3}$ annihilate Q_1.

Next observe that there are only a countable number of symplectic polynomials. Thus the generic set $\mathfrak{R}_0$ will include the countable intersection $\bigcap \tau_Q$, over the countable set of symplectic polynomials. The "exceptional periodic orbits" for any $dH^\# \in (\bigcap \tau_Q)$ will be the totality of the "exceptional periodic orbits" that $dH^\#$ has in connection with each symplectic polynomial Q. $\square$

Remark. Consider the discriminant Δ of the polynomial

$$\mu^{2n-2} + X_{2n-3}\mu^{2n-3} + \cdots + X_1\mu + X_0,$$

namely, in terms of the roots here denoted $\mu_1, \ldots, \mu_{2n-2}$,

$$\Delta = \prod_{1 \leq i < j < 2n-2} (\mu_i - \mu_j)^2.$$

By the fundamental theorem of symmetric functions, Δ is a rational polynomial; we write $\Delta(X_0, X_1, \ldots, X_{2n-3})$ in the $(2n - 2)$ indeterminants $(X_0, X_1, \ldots, X_{2n-3})$.

Clearly $\Delta(X_0, X_1, \ldots, X_{2n-3})$ is a symplectic polynomial since it vanishes for the characteristic coefficients $c_0, c_1, \ldots, c_{2n-3}$ of some matrix $T \in Sp(2n - 2, \mathbf{R})$ if and only if T has repeated eigenvalues. We then define the generic set τ_Δ corresponding to the single symplectic polynomial Δ. Thus we obtain the generic set $\mathfrak{R}$ in $\mathfrak{H}^k$ (for $k \geq 2$) found by Robinson [14]:

$$\mathfrak{R} = \{dH^\# \in \mathfrak{H}^k \mid \text{all but a countable number of periodic orbits of } dH^\# \text{ have distinct characteristic multipliers}\}.$$

Note that $\mathfrak{R} \supset \mathfrak{R}_0$ so each $dH^\# \in \mathfrak{R}_0$ has almost all of its periodic orbits of nondegenerate type.

3. Basic generic properties of periodic orbits of Hamiltonian systems. In this section we shall define precisely the sets $\mathfrak{S}_1$, $\mathfrak{S}_2$, and $\mathfrak{S}_3$ of Hamiltonian vector fields in terms of conditions on the periodic orbits. Then we shall establish that these classes are generic in $\mathfrak{H}^k$, for suitable $k = 1, 2, 3, \ldots, \infty$, on any symplectic manifold M of dimension $2n \geq 4$.

For $\mathfrak{S}_1$ we shall consider the arithmetic nature of the characteristic multipliers $\mu_2, \mu_3, \ldots, \mu_2^{-1}, \mu_3^{-1}, \ldots, \mu_n^{-1}$ of a periodic orbit γ.

For $\mathfrak{S}_2$ we examine how these characteristic multipliers change with the energy level h throughout the band of periodic orbits $\gamma(h)$ passing through $\gamma(0) = \gamma$. For $\mathfrak{S}_3$ we describe how the Poincaré map P_h deviates from linearity with radial displacements out from the periodic orbit γ, within the energy level $h = 0$.

Definition. Let $\mathfrak{H}^k$ be the Baire space of all Hamiltonian C^k-vector fields on a symplectic manifold M. Define the subsets, within $\mathfrak{H}^k$ for each fixed $k \geq 1$,

$$\mathfrak{S}_1 = \{dH^\# \in \mathfrak{H}^k \,|\, \text{all periodic orbits of } dH^\# \text{ have at most one of the characteristic multipliers } \mu_j \text{ (and } \mu_j^{-1}) \text{ that is a root of unity}\}.$$

and

$$\mathfrak{S}_2 = \{dH^\# \in \mathfrak{H}^k \,|\, \text{all but a countable number of periodic orbits of } dH^\# \text{ have distinct characteristic multipliers, some one of which satisfies } (d/dh)\mu_j(h) \neq 0 \text{ on the periodic orbit}\}.$$

THEOREM 3. *Let $\mathfrak{H}^k$ be the space of Hamiltonian vector fields on a symplectic manifold M. Then $\mathfrak{S}_1$ is generic in $\mathfrak{H}^k$, for each $k \geq 2$.*

Proof. Let γ be a periodic orbit for any Hamiltonian system $dH^\#$ in $\mathfrak{H}^k$, for $k \geq 2$. Let the Poincaré matrix dP_0 have the characteristic polynomial

$$F(\mu) \equiv \det|\mu I - dP_0| = \mu^{2n-2} + c_{2n-3}\mu^{2n-3} + \cdots + c_1\mu + c_0,$$

with the characteristic coefficients $c_{2n-3}, \ldots, c_1, c_0 = 1$. Similarly write the characteristic polynomial of $(dP_0)^\nu$, for each integral power $\nu = 2, 3, 4, \ldots$, in the format (not indicating any derivative)

$$F^{(\nu)}(\mu) \equiv \det|\mu I - (dP_0)^\nu|$$
$$= \mu^{2n-2} + c^{(\nu)}_{2n-3}\mu^{2n-3} + \cdots + c_1{}^{(\nu)}\mu + c_0{}^{(\nu)}.$$

Note that dP_0 has a ν-th root of unity among its eigenvalues $(\mu_2, \ldots,$

$\mu_n, \mu_2^{-1}, \ldots, \mu_n^{-1}$) if and only if the polynomial $F^{(\nu)}(\mu) = \det|\mu I - (dP_0)^\nu|$ vanishes at $\mu = 1$. This condition is just

$$F^{(\nu)}(1) = 1 + c_{2n-3}^{(\nu)} + \cdots + c_1^{(\nu)} + c_0^{(\nu)} = 0.$$

We observe that each of the coefficients $c_{2n-3}^{(\nu)}, \ldots, c_1^{(\nu)}, c_0^{(\nu)}$ is a symmetric polynomial in the roots $\mu_2^\nu, \ldots, \mu_n^\nu, \mu_2^{-\nu}, \ldots, \mu_n^{-\nu}$ of $F^{(\nu)}(\mu)$, and hence must be expressed as a rational polynomial in the elementary symmetric functions of the roots $\mu_2, \ldots, \mu_n, \mu_2^{-1}, \ldots, \mu_n^{-1}$ of $F(\mu)$. That is, there exist polynomials over the rational field $c_0^{(\nu)}(X_0, X_1, \ldots, X_{2n-3})$, $c_1^{(\nu)}(X_0, X_1, \ldots, X_{2n-3})$, $\ldots$, $c_{2n-3}^{(\nu)}(X_0, X_1, \ldots, X_{2n-3})$ in the indeterminants $(X_0, X_1, \ldots, X_{2n-3})$ such that these yield the values $c_0^{(\nu)}, c_1^{(\nu)}, \ldots, c_{2n-3}^{(\nu)}$ when evaluated at $c_0, c_1, \ldots, c_{2n-3}$. In this way we define the rational polynomial

$$Q^{(\nu)}(X_0, X_1, \ldots, X_{2n-3}) = 1 + c_{2n-3}^{(\nu)}(X_0, X_1, \ldots, X_{2n-3}) + \cdots + c_0^{(\nu)}(X_0, X_1, \ldots, X_{2n-3}).$$

Then the periodic orbit γ has a characteristic multiplier that is a ν-th root of unity if and only if $Q^{(\nu)}(X_0, X_1, \ldots, X_{2n-3})$ vanishes at the characteristic coefficients $(c_0, c_1, \ldots, c_{2n-3})$ of γ.

The polynomial $Q^{(\nu)}(X_0, X_1, \ldots, X_{2n-3})$ is a symplectic polynomial, as defined earlier, since not every symplectic matrix in $Sp(2n - 2, \mathbf{R})$ has an eigenvalue that is a ν-th root of unity. Thus the algebraic condition $Q^{(\nu)} = 0$, as evaluated for the characteristic coefficients of any periodic orbit γ, defines an algebraic variety, still called $Q^{(\nu)}$ when denoting a normal subset in $J^1(2n - 1)$. Moreover codim $Q^{(\nu)} \geq 1$, by Corollary 2 of the previous section.

Next we shall obtain a condition specifying that γ has a multiple ν-th root of unity among its characteristic multipliers $\mu_2, \mu_3, \ldots, \mu_n, \mu_2^{-1}, \mu_3^{-1}, \ldots, \mu_n^{-1}$. Of course $F(\mu)$ must have two ν-th roots of unity, say μ_j and μ_j^{-1}, if $F^{(\nu)}(1) = 0$. Hence we seek a condition that $F^{(\nu)}$ has $+1$ as a root of multiplicity of at least 3. That is, we demand that $dF^{(\nu)}/d\mu$ and $d^2F^{(\nu)}/d\mu^2$ both vanish at $\mu = 1$.

In order to express this demand in general terms, we define corresponding rational polynomials in the indeterminants $(X_0, X_1, \ldots, X_{2n-3})$ given by the formal derivatives

$$Q_1^{(\nu)}(X_0, \ldots, X_{2n-3}) = (2n-2) + (2n-3)c_{2n-3}^{(\nu)}(X_0, \ldots, X_{2n-3})$$
$$+ \cdots + 3c_3^{(\nu)} + 2c_2^{(\nu)} + c_1^{(\nu)},$$

and

$$Q_2^{(\nu)}(X_0, \ldots, X_{2n-3}) = (2n-2)(2n-3)$$
$$+ (2n-3)(2n-4)c_{2n-3}^{(\nu)}(X_0, \ldots, X_{2n-3})$$
$$+ \cdots + 6c_3^{(\nu)} + 2c_2^{(\nu)}.$$

Now define the rational polynomial

$$Q_4^{(\nu)}(X_0, \ldots, X_{2n-3}) = (Q_1^{(\nu)})^2 + (Q_2^{(\nu)})^2.$$

Then $Q_4^{(\nu)}(X_0, \ldots, X_{2n-3})$ is a symplectic polynomial whose vanishing on any characteristic coefficients $(c_0, c_1, \ldots, c_{2n-3})$ guarantees that γ has a 4-multiple ν-th root of unity among its characteristic multipliers, provided $Q^{(\nu)}(X_0, \ldots, X_{2n-3})$ also vanishes for γ.

But now we can apply the conclusions of Corollary 2 above that the algebraic variety $Q^{(\nu)} \cap Q_4^{(\nu)}$ designates a normal set in $J^1(2n-1)$ and furthermore codim $(Q^{(\nu)} \cap Q_4^{(\nu)}) \geq 2$. This follows from the fact that every symplectic matrix $T \in Sp(2n-2, \mathbf{R})$ that has $\mu = e^{2\pi i r/\nu}$ as an eigenvalue (with $\mu^\nu = e^{2\pi i r} = 1$) can be approximated by a symplectic matrix $\tilde{T}$ having only two eigenvalues ($e^{2\pi i r/\nu}$ and $e^{-2\pi i r/\nu}$) that are roots of unity. Such an approximation can be made in two steps; first approximate T by a symplectic matrix $\hat{T}$ "diagonalized within $Sp(2n-2), \mathbf{R})$," and then construct the required approximation $\tilde{T}$ from $\hat{T}$.

Now, following Taken's theorem above, we obtain a residual set $\mathfrak{S}_1^{(\nu)} \subset \mathfrak{H}^k$ corresponding to the normal subset $Q^{(\nu)} \cap Q_4^{(\nu)}$ of $J^1(2n-1)$. Clearly, by Corollary 1, each Hamiltonian system $dH^\# \in \mathfrak{S}_1^{(\nu)}$ has no periodic orbits having more than one characteristic multiplier μ_j (and also μ_j^{-1}) that is a ν-th root of unity.

Finally we define a residual set in $\mathfrak{H}^k$ by the countable intersection $\cap\, \mathfrak{S}_1^{(\nu)}$ for $\nu = 2, 3, 4, 5, 6, \ldots$. Suppose some Hamiltonian system $dH^\#$ in $\cap\, \mathfrak{S}_1^{(\nu)}$ had a periodic orbit γ with characteristic multipliers

μ_{j_1} as a ν_1-th root of unity, and also μ_{j_2} as a ν_2-th root of unity, for $\nu_1 \neq \nu_2$. Then both μ_{j_1} and μ_{j_2} can be considered to be $(\nu_1 \cdot \nu_2)$-th roots of unity. In this case γ would have a multiple $(\nu_1 \cdot \nu_2)$-th root of unity among its characteristic multipliers, which contradicts the definition of $\mathfrak{S}_1^{(\nu_1\nu_2)}$. Hence γ can have at most one characteristic multiplier among $(\mu_2, \ldots, \mu_n)$ that is a root of unity.

Thus we conclude that $\mathfrak{S}_1$, as defined above, must contain the residual set $\cap \mathfrak{S}_1^{(\nu)}$, and so $\mathfrak{S}_1$ is generic in $\mathfrak{H}^k$, for each $k \geq 2$. □

THEOREM 4. *Let $\mathfrak{H}^k$ be the space of Hamiltonian vector fields on a symplectic manifold M. Then $\mathfrak{S}_2$ is generic in $\mathfrak{H}^k$, for each $k \geq 3$.*

Proof. Take any Hamiltonian system $dH^\# \in \mathfrak{H}^k$, for fixed $k \geq 3$, on the symplectic manifold M. Assume $dH^\#$ lies in the Robinson generic set $\mathfrak{R}$ so that almost all periodic orbits of $dH^\#$ have distinct characteristic multipliers. Let γ be a periodic orbit of $dH^\#$, and assume that γ has distinct characteristic multipliers so that γ lies in a band of periodic orbits $\gamma(h)$ of $dH^\#$, where the energy level h is taken to be zero at $\gamma(0) = \gamma$. The Poincaré map P around γ has the restriction P_h on each energy level h, and the corresponding Poincaré matrix dP_h yields the characteristic polynomial

$$F_h(\mu) = \det_4 |\mu I - dP_h| = \mu^{2n-2} + c_{2n-3}\mu^{2n-3} + \cdots + c_1\mu + c_0.$$

Here we consider the characteristic coefficients $c_{2n-3}(h), \ldots, c_1(h) \equiv 1$ as C^k-functions of the energy level h, and the roots $\mu_2(h), \ldots, \mu_n(h)$, $\mu_2^{-1}(h), \ldots, \mu_n^{-1}(h)$ are also distinct (complex) C^k-functions for h near zero. Note that the trace of dP_h is just the sum of the eigenvalues, so

$$\begin{aligned}\operatorname{Tr}(dP_h) &= \mu_2(h) + \cdots + \mu_n(h) + \mu_2^{-1}(h) + \cdots + \mu_n^{-1}(h)\\ &= -c_{2n-3}(h)\end{aligned}$$

Consider the 2-jet $P^{(2)}$ in $J^2(2n-1)$ which is embedded in $M_0{}^2(2n-1)$, the real linear space of all 2-jets of maps of $(\mathbf{R}^{2n-1}, 0)$ into itself. We shall find a polynomial in $M_0{}^2(2n-1)$ whose vanishing expresses the condition $dc_{2n-3}(h)/dh = 0$ at $h = 0$.

Take any linear transformation $S = (s_{ij})$ of $\mathbf{R}^{2n-2}$ into itself. Then

$$\operatorname{Tr} S = s_{11} + s_{22} + \cdots + s_{n-2,n-2}$$

is a polynomial on the matrix space $M_0{}^1(2n - 2)$. If $S(h)$ depends differentiably on h, then

$$\frac{d}{dh}[\mathrm{Tr}\, S(h)]_{h=0} = \frac{ds_{11}}{dh} + \cdots + \frac{ds_{2n-2,2n-2}}{dh}$$

is a polynomial $\hat{Q}$ defining an algebraic variety $[\hat{Q}]$ in $M_0{}^2(2n - 1)$. Since the most general parameter-symplectic automorphism of $\mathbf{R} \times (\mathbf{R}^{n-1} \oplus \mathbf{R}^{n-1})$ is derived from a symplectic map on each energy level h, preserving the values of h, the intersection $Q = [\hat{Q}] \cap J^2(2n - 1)$ defines a normal subset of $J^2(2n - 1)$, according to the notations and discussions leading to Corollary 2 of the previous section. Also codim $Q \geq 1$.

Take the corresponding residual set $\tau_Q \subset \mathfrak{H}^k$ following Taken's theorem. Then for $dH^\# \in \tau_Q \cap \mathfrak{R}$ only a countable collection of exceptional periodic orbits of $dH^\#$ will have their 2-jets $P^{(2)}$ annihilating the polynomial $\hat{Q}$. This means that

$$\frac{d}{dh}[\mathrm{Tr}\,(dP_h)]_{h=0} = 0$$

obtains only for the countable collection of exceptional periodic orbits of $dH^\# \in \tau_Q \cap \mathfrak{R}$. Hence, excluding the exceptional periodic orbits of $dH^\#$,

$$\frac{d}{dh}[\mathrm{Tr}\,(dP_h)]_{h=0} = \frac{d}{dh}[\mu_2(h) + \cdots + \mu_n(h) + \mu_2{}^{-1}(h) + \cdots + \mu_n{}^{-1}(h)]_{h=0} \neq 0.$$

But this implies that some characteristic multiplier $\mu_j(h)$ of dP_h satisfies

$$\frac{d}{dh}[\mu_j(h)]_{h=0} \neq 0.$$

Hence $\mathfrak{S}_2$ contains $\tau_Q \cap \mathfrak{R}$, and so $\mathfrak{S}_2$ is generic in $\mathfrak{H}^k$, for each $k \geq 3$. □

Remark. From its definition $\mathfrak{S}_2 \subset \mathfrak{R}$, but we often explicitly indicate the generic set $\mathfrak{R}$ even when this notation is redundant. To illustrate the significance of the generic properties $\mathfrak{S}_1$ and $\mathfrak{S}_2$ let us consider the generic set $\mathfrak{R} \cap \mathfrak{S}_1 \cap \mathfrak{S}_2 \subset \mathfrak{H}^k$, for any $k \geq 3$, of Hamiltonian systems on a symplectic manifold M. Let γ be a periodic orbit of any $dH^\# \in \mathfrak{R} \cap \mathfrak{S}_1 \cap \mathfrak{S}_2$ and assume that γ is an elliptic orbit with distinct characteristic multipliers $\mu_2, \ldots, \mu_n, \mu_2^{-1}, \ldots, \mu_n^{-1}$. In this case γ lies in a band of elliptic periodic orbits $\gamma(h)$ of $dH^\#$, say $\gamma(0) = \gamma$, and each has distinct characteristic multipliers $\mu_2(h), \ldots, \mu_n(h), \mu_2^{-1}(h), \ldots, \mu_n^{-1}(h)$.

Then $\gamma(0)$, or else some close approximation $\gamma(h_1)$ for h_1 near zero, has some $d\mu_j/dh \neq 0$. But this means that some close approximation $\gamma(h_2)$, for h_2 near h_1, must have a characteristic multiplier μ_j that is a root of unity. Moreover $\gamma(h_2)$ does not have any other root of unity (excepting μ_j and μ_j^{-1}) among its characteristic multipliers.

Thus allowing some possible arbitrarily close approximation, we can assume that γ has distinct characteristic frequencies $\omega_2, \ldots, \omega_n$ (mod 1) with just one of these a rational number and all the others irrational. That is, after a suitable small shift in the energy level, we can assume that any elliptic orbit γ with distinct characteristic frequencies $\omega_2, \ldots, \omega_n$ has precisely one rational characteristic frequency.

Finally we turn to the definition of the generic set $\mathfrak{S}_3 \subset \mathfrak{H}^k$, for each $k \geq 4$, on the symplectic $2n$-manifold M. In order to obtain an intrinsic geometric condition that the Poincaré map P_h around an elliptic orbit γ of $dH^\# \in \mathfrak{H}^k$ is suitably nonlinear, we must give a precise description of the Birkhoff normal form for the symplectic map P_0.

Let $E(2n-2) \subset Sp(2n-2, \mathbf{R})$ be the set of all elliptic symplectic matrices having distinct eigenvalues $\mu_j = e^{2\pi i\omega_j}$ and $\mu_j^{-1} = e^{-2\pi i\omega_j}$ for $j = 2, 3, \ldots, n$. Note that $E(2n-2)$ is an open subset of $Sp(2n-2, \mathbf{R})$, and its boundary is contained as part of the set Δ where the discriminant symplectic polynomial $\Delta(X_0, X_1, \ldots, X_{2n-3})$ vanishes on $Sp(2n-2, \mathbf{R})$. The linear diagonalization of any such matrix $T \in E(2n-2)$ is a classical result [12], namely there exists $F \in Sp(2n-2, \mathbf{R})$ such that

$$FTF^{-1} = \operatorname{diag}\left\{\begin{pmatrix} \cos 2\pi\omega_j & -\sin 2\pi\omega_j \\ \sin 2\pi\omega_j & \cos 2\pi\omega_j \end{pmatrix}\right\} \qquad \text{for } j = 2, \ldots, n.$$

Moreover this symplectic canonical form for T can be made unique by ordering the 2×2 rotation factors according to the frequencies $0 < \omega_j$

< 1. Also the diagonalizing matrix F is then unique, up to multiplication by a diagonalized rotation, say of the form

$$F_1 = \operatorname{diag}\left\{\begin{pmatrix} \cos 2\pi f_j & -\sin 2\pi f_j \\ \sin 2\pi f_j & \cos 2\pi f_j \end{pmatrix}\right\} F.$$

This follows from the fact that the centralizer of any fixed nontrivial rotation of the plane $\mathbf{R}^2$, within the group $Sp(2, \mathbf{R})$, is just the rotation group $SO(2, \mathbf{R})$.

The Birkhoff normal form concerns the "nonlinear diagonalization" of any nonlinear symplectic map Φ within the group $Sp^k(2n - 2)$ of germs of symplectic C^k-maps of $(\mathbf{R}^{n-1} \oplus \mathbf{R}^{n-1}, 0)$ into itself, for fixed $k \geq 4$. Hence, we seek a diagonalized canonical form for a map

$$\Phi\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X(x, y) \\ Y(x, y) \end{pmatrix} \qquad \text{in } Sp^k(2n - 2)$$

under an inner automorphism of $Sp^k(2n - 2)$, that is, under the similarity relation $\Psi\Phi\Psi^{-1}$ for some map

$$\Psi\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \qquad \text{in } Sp^k(2n - 2).$$

The l-jets of maps of $Sp^k(2n - 2)$ form the symplectic jet space $J_s{}^l(2n - 2)$, for $1 \leq l \leq k$. We can consider $J_s{}^l(2n - 2)$ as a Lie subgroup of $J^l(2n - 1)$, in fact a normal subgroup. Moreover, just as in the lemma of the preceding section, $J_s{}^l(2n - 2)$ is a closed analytic submanifold in $J^l(2n - 1)$. Each lower order jet space, for instance $J_s{}^1(2n - 2) = Sp(2n - 2, \mathbf{R})$, has an invariant natural embedding in $J_s{}^l(2n - 2)$ and hence in $J^l(2n - 1)$. In particular the set $E(2n - 2)$ can also determine an open subset of $J^l(2n - 1)$, when consider as a restriction on 1-jets only.

The Jacobian matrix or 1-jet $\Phi^{(1)}$ of Φ is required to be an elliptic symplectic matrix with distinct eigenvalues, that is, $\Phi^{(1)} \in E(2n - 2)$. But further, we shall require $\Phi^{(1)}$ have characteristic frequencies $\omega_2, \ldots, \omega_n$ that are "linearly independent over the small integers," as indicated below.

Definition. Let $E_a(2n - 2) \subset E(2n - 2)$ consist of all elliptic symplectic matrices T with eigenvalues $\mu_2 = e^{2\pi i \omega_2}, \ldots, \mu_n = e^{2\pi i \omega_n}$, $\mu_2^{-1}, \ldots, \mu_n^{-1}$ such that:

$$\mu_2^{\alpha_2}\mu_3^{\alpha_3} \cdots \mu_n^{\alpha_n} = 1, \text{ or equivalently,}$$

$$\alpha_2\omega_2 + \alpha_3\omega_3 + \cdots + \alpha_n\omega_n = 0 \pmod 1$$

is impossible for any choice of integers $\alpha_2, \ldots, \alpha_n$ satisfying

$$0 < \sum_{j=2}^{n} |\alpha_j| \leq 5.$$

Remarks. Note that the condition of independence forces the eigenvalues $\mu_2, \ldots, \mu_n, \mu_2^{-1}, \ldots, \mu_n^{-1}$ to be distinct, and in particular none of these is $+1$ or -1. In fact, by a correct choice of μ_j or μ_j^{-1}, we can demand that $0 < \omega_j < 1/2$ for $j = 2, \ldots, n$ be distinct frequencies.

Also $E_a(2n - 2)$ is an open and dense subset of $E(2n - 2)$. In fact, $E_a(2n - 2)$ is obtained by deleting a finite collection of real analytic subvarieties from $E(2n - 2)$. This can be verified by noting that each Re μ_j, $Im\mu_j$, and also ω_j, are real analytic functions of the entries of any matrix $T \in E(2n - 2)$. In addition there are only a finite number of independence conditions imposed. Thus the boundary of $E_a(2n - 2)$ in $E(2n - 2)$ consists of a real analytic set with empty interior in $E(2n - 2)$. Of course this real analytic set may not be closed in the closure $\overline{E}(2n - 2) \subset Sp(2n - 2, \mathbf{R})$, and special attention must be paid to the boundary of $\overline{E}(2n - 2)$.

Finally note that $E_a(2n - 2)$ is an invariant subset of $E(2n - 2)$, which is itself invariant under every inner automorphism of $Sp(2n - 2, \mathbf{R})$. In this way $E_a(2n - 2)$ can determine an open invariant subset of $J_s^{\,l}(2n - 2)$, and even in $J^l(2n - 1)$, for every jet space with $l \geq 1$.

THEOREM (BIRKHOFF). *Let $\Phi \in Sp^k(2n - 2)$ for fixed $k \geq 4$ have the 1-jet $\Phi^{(1)} \in E_a(2n - 2)$ which can be diagonalized to the symplectic canonical form*

$$\Phi^{(1)} \sim \operatorname{diag}\left\{\begin{pmatrix} \cos 2\pi\omega_j & -\sin 2\pi\omega_j \\ \sin 2\pi\omega_j & \cos 2\pi\omega_j \end{pmatrix}\right\}.$$

Then there exists a map $\Psi \in Sp^k(2n-2)$

$$\Psi : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad or \quad \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \Psi\begin{pmatrix} x \\ y \end{pmatrix}$$

such that $\Psi\Phi\Psi^{-1}$ *has the Birkhoff normal form, written in terms of the symplectic polar coordinates*

$$u_j = \frac{\xi_j^2 + \eta_j^2}{2}, \quad \theta_j = \arc\tan \eta_j/\xi_j \qquad j = 2, \ldots, n,$$

$$\Psi\Phi\Psi^{-1} : (u, \theta) \to (U, \Theta)$$

where

$$U_j = u_j + \tilde{U}_j, \quad and \quad \tilde{U}_j = O(\|u\|^2)$$

$$\Theta_j = \theta_j + 2\pi\omega_j + \sum_{l=2}^{n} c_{jl} u_l + \tilde{\Theta}_j, \quad and \quad \tilde{\Theta}_j = O(\|u\|^2).$$

Moreover the coefficient matrices (ω_j) *and* (c_{jl}) *are uniquely determined by the 3-jet* $\Phi^{(3)}$*. Furthermore, if*

$$\Phi^{(1)} = \operatorname{diag}\left\{\begin{pmatrix} \cos 2\pi\omega_j & -\sin 2\pi\omega_j \\ \sin 2\pi\omega_j & \cos 2\pi\omega_j \end{pmatrix}\right\},$$

then we can require $\Psi^{(1)} =$ *Identity and in this case* $\Psi^{(3)}$ *is also uniquely determined by* $\Phi^{(3)}$.

Remarks. The theorem of Birkhoff is not usually presented in the generality asserted here. [3 p. 822] In the standard statement the map $\tilde{\Phi} \in Sp^k(2n-2)$ already has the diagonalized 1-jet

$$\tilde{\Phi}^{(1)} = \operatorname{diag}\left\{\begin{pmatrix} \cos 2\pi\omega_j & -\sin 2\pi\omega_j \\ \sin 2\pi\omega_j & \cos 2\pi\omega_j \end{pmatrix}\right\}$$

and the diagonalizing map $\tilde{\Psi} \in Sp^k(2n-2)$ is normalized by

$$\tilde{\Psi}^{(1)} = I.$$

In this case the Birkhoff reduction shows that (ω_j), (c_{jl}), and $\tilde{\Psi}^{(3)}$ are uniquely determined by the 3-jet $\tilde{\Phi}^{(3)}$.

Indeed, (ω_j) are the characteristic frequencies of $\tilde{\Phi}^{(1)}$ and the matrix (c_{jl}) is obtained by the following procedure.

There exist certain universal polynomials B_{jl}, called the Birkhoff polynomials in σ-indeterminants (where σ is the number of components of the nonlinear terms in the 3-jet space $M_0{}^3(2n-2)$), such that (c_{jl}) are just B_{jl} evaluated on the coefficients of the nonlinear terms of $\tilde{\Phi}^{(3)}$. The Birkhoff polynomials B_{jl} have coefficients in the field of rational functions of $(2n-2)$ indeterminants, over the base rational field, and these coefficients are to be evaluated at the $(2n-2)$ values $\cos 2\pi\omega_j$ and $\sin 2\pi\omega_j$.

In the next lemma we demonstrate the uniqueness of the Birkhoff coefficients (ω_j) and (c_{jl}) for any map $\Phi \in Sp^k(2n-2)$, with the 1-jet $\Phi^{(1)} \in E_a(2n-2)$ not necessarily in the symplectic canonical form.

LEMMA. *Fix $\Phi_0 \in Sp^k(2n-2)$ with $\Phi_0{}^{(1)} \in E_a(2n-2)$. Then there exists a neighborhood V^3 of $\Phi_0{}^{(3)}$ in $J_s{}^3(2n-2)$ such that: for each map $\Phi \in Sp^k(2n-2)$ with $\Phi^{(3)} \in V^3$*

1) *The Birkhoff coefficients (ω_j) and (c_{jl}) of Φ are uniquely determined, and*
2) *The matrix entries of (ω_j) and (c_{jl}) are real analytic functions on V^3.*

Proof. We first show that the Birkhoff coefficients (ω_j) and (c_{jl}) are uniquely determined for Φ_0, or for any Φ near Φ_0, whether or not the 1-jet is assumed to be in the symplectic canonical form as a diagonalized matrix of rotations.

Take any Φ with $\Phi^{(3)}$ near $\Phi_0{}^{(3)}$ in $J_s{}^{(3)}(2n-2)$, so $\Phi^{(1)} \in E_a(2n-2)$. Then there exists $F \in Sp(2n-2)$ so

$$F\Phi^{(1)}F^{-1} = \text{diag}\left\{\begin{pmatrix} \cos 2\pi\omega_j & -\sin 2\pi\omega_j \\ \sin 2\pi\omega_j & \cos 2\pi\omega_j \end{pmatrix}\right\},$$

with (ω_j) uniquely determined as the characteristic frequencies of $\Phi^{(1)}$, as displayed in this symplectic canonical form. Any other diagonalizing matrix $F_1 \in Sp(2n - 2)$ that places $\Phi^{(1)}$ in symplectic canonical form, must itself be related to F by

$$F_1 = \text{diag}\left\{\begin{pmatrix} \cos 2\pi f_j & -\sin 2\pi f_j \\ \sin 2\pi f_j & \cos 2\pi f_j \end{pmatrix}\right\} F.$$

The map $\tilde{\Phi} = F\Phi F^{-1}$ can thus be transformed to its Birkhoff normal form by a unique $\tilde{\Psi} \in Sp^4(2n - 2)$ with the normalization $\tilde{\Psi}^{(1)} = I$. That is, $\tilde{\Psi}\tilde{\Phi}\tilde{\Psi}^{-1}$ has the required diagonalized form in symplectic polar coordinates (u, θ) according to

$$u_j \to U_j = u_j + \tilde{U}_j$$

$$\theta_j \to \Theta_j = \theta_j + 2\pi\omega_j + \Sigma\, c_{jl}u_l + \tilde{\Theta}_j \qquad \text{for } j = 2, \ldots, n.$$

Also the Birkhoff matrix (c_{jl}) can be computed by evaluating the Birkhoff polynomials B_{jl} on the data of $\tilde{\Phi}^{(3)}$.

To relate this Birkhoff normal form for $\tilde{\Phi}$ to the given map Φ, we write

$$\tilde{\Psi}\tilde{\Phi}\tilde{\Psi}^{-1} = \tilde{\Psi}F\Phi F^{-1}\tilde{\Psi}^{-1} = \Psi\Phi\Psi^{-1},$$

with $\Psi = \tilde{\Psi}F \in Sp^4(2n - 2)$. This shows that Φ can be placed in Birkhoff normal form, by the diagonalizing map Ψ, so as to display precisely the same Birkhoff coefficient matrices (ω_j) and (c_{jl}) as for $\tilde{\Phi} = F\Phi F^{-1}$.

Next we show that (ω_j) and (c_{jl}) are uniquely determined by $\Phi^{(3)}$ in the sense that there is a unique Birkhoff normal form obtained from Φ and this involves only the data of $\Phi^{(3)}$. Certainly (ω_j) are just the characteristic frequencies of $\Phi^{(1)}$ and so these are intrinsically specified, but we must show also that (c_{jl}) do not depend on the choice of the matrix $F \in Sp(2n - 2)$ or the diagonalizing map $\Psi \in Sp^k(2n - 2)$.

Suppose $\Psi_1 \in Sp^k(2n - 2)$ also diagonalizes Φ, so $\Psi_1 \Phi \Psi_1^{-1}$ is in Birkhoff normal form. Write $\tilde{\Phi} = F\Phi F^{-1}$ so $\tilde{\Phi}^{(1)}$ is in symplectic canonical form, and further

$$\Psi F^{-1}\tilde{\Phi} F \Psi^{-1} = \Psi \Phi \Psi^{-1}$$

and

$$\Psi_1 F^{-1}\tilde{\Phi} F \Psi_1^{-1} = \Psi_1 \Phi \Psi_1^{-1}$$

are both in some Birkhoff normal forms. Define the maps $\hat{\Psi} = \Psi F^{-1}$ and $\hat{\Psi}_1 = \Psi_1 F^{-1}$, and then these two Birkhoff normal forms are given by $\hat{\Psi}\tilde{\Phi}\hat{\Psi}^{-1}$ and $\hat{\Psi}_1 \tilde{\Phi} \hat{\Psi}_1^{-1}$. Now the 1-jets $\hat{\Psi}^{(1)}$ and $\hat{\Psi}_1^{(1)}$ commute with the symplectic canonical matrix $\tilde{\Phi}^{(1)}$, so necessarily they are each diagonal rotation matrices

$$\hat{\Psi}^{(1)} = R \quad \text{and} \quad \hat{\Psi}_1^{(1)} = R_1.$$

Then we can define $\tilde{\Psi} = R^{-1}\hat{\Psi}$ and $\tilde{\Psi}_1 = R_1^{-1}\hat{\Psi}_1$ so $\tilde{\Psi}^{(1)} = \tilde{\Psi}_1^{(1)} = I$, and moreover the two Birkhoff normal forms for Φ are represented by

$$R\tilde{\Psi}\tilde{\Phi}\tilde{\Psi}^{-1}R^{-1} \quad \text{and} \quad R_1\tilde{\Psi}_1\tilde{\Phi}\tilde{\Psi}_1^{-1}R_1^{-1}.$$

But any Birkhoff normal form is unchanged by any composition with a linear rotation of the symplectic polar coordinates (u_j, θ_j), so we conclude that the two Birkhoff normal forms for Φ are also represented by the maps

$$\tilde{\Psi}\tilde{\Phi}\tilde{\Psi}^{-1} \quad \text{and} \quad \tilde{\Psi}_1\tilde{\Phi}\tilde{\Psi}_1^{-1}.$$

That is, these two maps display the two possible choices of the Birkhoff coefficient matrix (c_{jl}).

But here the classical uniqueness theorem of Birkhoff applies to show that these two Birkhoff normal forms of $\tilde{\Phi}$ must be the same, in that they display the same Birkhoff coefficient matrices (ω_j) and (c_{jl}). In fact, (c_{jl}) can be calculated by evaluating the Birkhoff polynomials B_{jl} on the data of $\tilde{\Phi}^{(3)}$. Hence (ω_j) and (c_{jl}) are uniquely specified by $\Phi^{(3)}$, even though there is a choice of matrix $F \in Sp(2n - 2)$ used in diagonalizing $\Phi^{(1)}$.

Now we permit Φ to vary near Φ_0, as long as $\Phi^{(3)}$ remains in a suitable neighborhood V^3 of $\Phi_0^{(3)}$ in $J_s^3(2n-2)$. Then the matrices (ω_j) and (c_{jl}) are functions of $\Phi^{(3)}$ in V^3, say with values $(\omega_j(0))$ and $(c_{jl}(0))$ at $\Phi_0^{(3)}$.

Let $F_0 \in Sp(2n-2)$ place $\Phi_0^{(1)}$ in symplectic canonical form

$$F_0\Phi_0^{(1)}F_0^{-1} = \operatorname{diag}\left\{\begin{pmatrix}\cos 2\pi\omega_j(0) & -\sin 2\pi\omega_j(0)\\ \sin 2\pi\omega_j(0) & \cos 2\pi\omega_j(0)\end{pmatrix}\right\}.$$

There exists a neighborhood V^1 of $\Phi_0^{(1)}$ in $Sp(2n-2)$, and a definite choice of matrix $F \in Sp(2n-2)$ for each $\Phi^{(1)} \in V^1$ such that

$$F\Phi^{(1)}F^{-1} = \operatorname{diag}\left\{\begin{pmatrix}\cos 2\pi\omega_j & -\sin 2\pi\omega_j\\ \sin 2\pi\omega_j & \cos 2\pi\omega_j\end{pmatrix}\right\},$$

and furthermore F can be required to depend analytically on $\Phi^{(1)}$ in V^1. That is, F defines an analytic map

$$F: V^1 \to Sp(2n-2): \Phi^{(1)} \to F \quad \text{with} \quad \Phi_0^{(1)} \to F_0.$$

This is a classical result on the diagonalization of matrices having distinct eigenvalues.

Now take V^1 suitably small in $E_a(2n-2)$ and define the open set $V^3 \subset J_s^3(2n-2)$ as specified by the condition

$$\Phi^{(3)} \in V^3 \quad \text{in case} \quad \Phi^{(1)} \in V^1.$$

Since the Birkhoff coefficients (c_{jl}) are rational functions of $F\Phi^{(3)}F^{-1} = \tilde{\Phi}^{(3)}$, they are analytic functions of $\Phi^{(3)}$ once F has been specified analytically on V^1. Thus $\Phi^{(3)} \to (c_{jl})$ is an analytic map for $\Phi^{(3)} \in V^3$ and (c_{jl}) in the space of real $(n-1) \times (n-1)$ matrices. Of course, $\omega_j = (1/2\pi i) \ln \mu_j$ are also real analytic functions on V^3 with values in the interval $(0, 1)$. □

COROLLARY. *The Birkhoff normal form provides an analytic map*

$$B: E_a^3(2n-2) \to \mathbf{R}^{n-1} \times M_0^1(n-1)$$

defined by

$$\Phi^{(3)} \to (\omega_j) \times (c_{jl}).$$

Here $E_a{}^3(2n - 2)$ *is the open subset of* $J_s{}^3(2n - 2)$ *defined by the condition* $\Phi^{(1)} \in E_a(2n - 2)$, *and* $M_0{}^1(n - 1)$ *is the linear space of all real* $(n - 1) \times (n - 1)$ *matrices.*

In this sense det (c_{jl}) is a real analytic function on $E_a{}^3(2n - 2)$. Thus the locus

$$\det (c_{jl}) = 0$$

is a real analytic variety in $J_s{}^3(2n - 2)$ which is invariant under all inner automorphisms. Also another such invariant is given by the *twist coefficient.*

Definition. Consider an elliptic periodic orbit γ of a Hamiltonian system $dH^\# \in \mathfrak{H}^k$, for $k \geq 4$, and let the Poincaré map P_0 of γ have a 1-jet $P_0{}^{(1)} \in E_a(2n - 2)$. Then (ω_j) and (c_{jl}) are defined as the Birkhoff coefficients of γ, and the twist coefficient of γ is

$$Tw(c_{jl}) = \prod_{j=2}^{n} c_{jj} = c_{22} \cdot c_{33} \cdots c_{nn}.$$

Definition. Let $\mathfrak{H}^k$ be the Baire space of all Hamiltonian C^k-vector fields on a symplectic manifold M. Define the subset of $\mathfrak{H}^k$, for each fixed $k \geq 4$,

$$\mathfrak{S}_3 = \{dH^\# \in \mathfrak{H}^k \mid \text{all but a countable number of the elliptic periodic orbits of } dH^\# \text{ have a twist coefficient } Tw(c_{jl}) \neq 0\}.$$

THEOREM 5. *Let* $\mathfrak{H}^k$ *be the space of Hamiltonian vector fields on a symplectic manifold M. Then* $\mathfrak{S}_3$ *is generic in* $\mathfrak{H}^k$, *for each* $k \geq 4$.

Proof. Take $\mathfrak{H}^k$, for fixed $k \geq 4$, and pick a generic subset, say $\mathfrak{R} \subset \mathfrak{H}^k$, corresponding to the discriminant symplectic polynomial $\Delta(X_0, X_1, \ldots, X_{2n-3})$. Then for $dH^\# \in \mathfrak{R}$ all but a countable number of periodic orbits of $dH^\#$ have distinct characteristic multipliers $\mu_2, \ldots, \mu_n, \mu_2{}^{-1}, \ldots, \mu_n{}^{-1}$. Hence among all degenerate and nondegenerate elliptic orbits of $dH^\# \in \mathfrak{R}$ (all orbits with all $|\mu_j| = 1$), excepting a countable set of orbits, each elliptic orbit has a 1-jet $P_0^{(1)}$ in $E(2n - 2) \subset J_s^{1}(2n - 2)$. Thus we need only consider elliptic orbits of $dH^{\#} \in \mathfrak{N}$ that have $P_0^{(1)} \in E(2n - 2)$.

Next we wish to restrict the study to elliptic orbits of $dH^{\#}$ with $P_0^{(1)}$ in the open subset $E_a(2n - 2) \subset E(2n - 2)$. For this purpose we must try to make $P_0^{(1)}$ avoid a finite set of analytic varieties of the form

$$A(\alpha_2, \ldots, \alpha_n): \mu_2^{\alpha_2}\mu_3^{\alpha_3} \cdots \mu_n^{\alpha_n} = 1,$$

for a set of integers $(\alpha_2, \ldots, \alpha_n)$ satisfying $0 < \Sigma^n_{j=2} |\alpha_j| \leq 5$. In the open set $E(2n - 2)$ each of $A(\alpha_2, \ldots, \alpha_n)$ describes an analytic variety, but its behavior in the closure $\overline{E}(2n - 2)$ is unclear. Hence we proceed by a method of exhausting the open set $E(2n - 2)$ by a sequence of closed subdomains.

For each fixed integer $N = 1, 2, 3, \ldots$ consider the open subset $E^{(N)}(2n - 2)$ of $E(2n - 2)$ where the discriminant polynomial $\Delta > 1/N$. Consider the closure $\overline{E^{(N)}}(2n - 2)$ in $E(2n - 2)$. Define $\mathfrak{S}_{3a}{}^{(N)} \subset \mathfrak{N} \subset \mathfrak{H}^k$ by

$$\mathfrak{S}_{3a}{}^{(N)} = \{dH^{\#} \in \mathfrak{H}^k \mid \text{all but a countable number of the elliptic periodic orbits of } dH^{\#}\text{, which have 1-jet } P_0^{(1)} \in \overline{E^{(N)}}(2n - 2) \text{ must, in fact, have } P_0^{(1)} \in E_a(2n - 2)\}.$$

By the remarks following Corollary 2 of the preceding section, $\mathfrak{S}_{3a}{}^{(N)}$ is generic in $\mathfrak{H}^k$. Now define the generic set $\mathfrak{S}_{3a} = \bigcap_{N=1} \mathfrak{S}_{3a}{}^{(N)}$ for $N \geq 1$. Then for each $dH^{\#} \in \mathfrak{S}_{3a}$ there are only a countable number of periodic orbits that have a 1-jet $P_0^{(1)} \in \overline{E}(2n - 2) - E_a(2n - 2)$. That is, upon discarding a countable number of exceptional elliptic periodic orbits of $dH^{\#} \in \mathfrak{S}_{3a}$, every other elliptic periodic orbit has a 1-jet $P_0^{(1)}$ in the open set $E_a(2n - 2)$.

For each periodic orbit γ of $dH^{\#} \in \mathfrak{S}_{3a}$ having 1-jet $P_0^{(1)} \in E_a(2n - 2)$, there is a well-defined twist coefficient $Tw(c_{jl})$. Furthermore the locus $Tw(c_{jl}) = 0$ is a real analytic variety in $E_a{}^3(2n - 2) \subset J_s{}^3(2n - 2)$, as asserted in the above corollary. We now define $\mathfrak{S}_{3t}$ within $\mathfrak{N} \subset \mathfrak{H}^k$ by

$$\mathfrak{S}_{3t} = \{dH^{\#} \in \mathfrak{H}^k \mid \text{all but a countable number of the elliptic periodic orbits of } dH^{\#}\text{, which have } P_0^{(1)} \in E_a(2n - 2)\text{, must have } Tw(c_{jl}) \neq 0\}.$$

Using the method of exhausting $E_a^3(2n - 2)$ by closed subdomains (say, defined by complements of neighborhoods of the analytic sets $A(\alpha_2, \ldots, \alpha_n)$ in $E(2n - 2)$, and by the condition $\Delta \geq 1/N$), we conclude that $\mathfrak{S}_{3t}$ is generic in $\mathfrak{H}^k$ for each $k \geq 4$.

Finally note that $\mathfrak{S}_3$ contains the generic set $\mathfrak{R} \cap \mathfrak{S}_{3a} \cap \mathfrak{S}_{3t}$, and so $\mathfrak{S}_3$ is generic in $\mathfrak{H}^k$ for each $k \geq 4$. □

4. Existence of periodic orbits and solenoids for Hamiltonian systems. In the preceding two sections we have considered generic subsets of the Baire space $\mathfrak{H}^k$ of all Hamiltonian C^k-vector fields, for $4 \leq k \leq \infty$, on any symplectic $2n$-manifold M, for $2n \geq 4$. In particular, we proved that

$$\mathfrak{M}_\Sigma = \mathfrak{R} \cap \mathfrak{S}_0 \cap \mathfrak{S}_1 \cap \mathfrak{S}_2 \cap \mathfrak{S}_3$$

is a generic subset of each such space $\mathfrak{H}^k$. Using the generic condition $\mathfrak{S}_0$, that all critical points are generic, we shall prove the existence of elliptic orbits in the vicinity of an elliptic critical point, as in Theorem 6 below. As remarked earlier, the generic conditions $\mathfrak{R} \cap \mathfrak{S}_1 \cap \mathfrak{S}_2$ guarantee that near any elliptic orbit there is another elliptic orbit which will have distinct characteristic frequencies $\omega_2, \ldots, \omega_n$ (mod 1), precisely one of which is rational. The twist condition $\mathfrak{S}_3$ will prove crucial in our main existence Theorem 7 where we construct a sequence of long period elliptic orbits.

In this section we demonstrate the existence of elliptic periodic orbits for each Hamiltonian system $dH^\# \in \mathfrak{M}_\Sigma$. Furthermore, we show that a suitably selected sequence of elliptic orbits converges to a preselected solenoid Σ_a, which occurs as a minimal set of the Hamiltonian dynamical system $dH^\#$. In our method we first produce an initial elliptic periodic orbit γ_0 near an elliptic critical point of $dH^\#$ (note: for this step alone we need $dH^\# \in \mathfrak{S}_0$ with M compact). Then we produce another elliptic orbit γ_1 of $dH^\#$ encircling a tubular neighborhood of γ_0. We then proceed by an inductive argument to obtain the required sequence of elliptic orbits converging to Σ_a. The details of the proof are given in our Principal Theorem.

We now proceed to locate the initial elliptic orbit γ_0 near an elliptic critical point Q_0 of $dH^\# \in \mathfrak{S}_0$, in accord with the classic theorem of Liapunov as modified slightly for our theory.

THEOREM 6. *Let the Hamiltonian system* $dH^\# \in \mathfrak{H}^2$ *have a generic elliptic critical point* Q_0. *Then in each neighborhood* N *of* Q_0 *in* M, *there exists an elliptic periodic orbit* γ_0 *of* $dH^\#$ *with distinct characteristic frequencies* $0 < \omega_2, \ldots, \omega_n < 1/2$.

Proof. We consider the Hamiltonian system $dH^\#$ in local canonical coordinates

$$z = \begin{pmatrix} x \\ y \end{pmatrix}$$

centered about Q_0 in M, wherein

$$H = \frac{1}{2} z'Sz + f(z)$$

with $S' = S$ and $f(z)$ of higher order than quadratic at $z = 0$. Then the Hamiltonian differential system has the form

$$dH^\#: \qquad \dot{z} = Az + g(z),$$

where $A = JS$ is a Hamiltonian matrix and $g(z) = J(\partial f/\partial z)'$ in C^2 satisfies $g(0) = 0$, $\partial g/\partial z(0) = 0$.

For convenience we introduce new local coordinates $w = z/\epsilon$ in terms of a "small scale factor" $\epsilon > 0$. Then the differential system in coordinates (w) becomes

$$(*) \qquad \dot{w} = Aw + h(w, \epsilon)$$

where $h(w, \epsilon) = (1/\epsilon)g(\epsilon w)$. It is easy to check that $h(w, \epsilon)$ is of class C^1 in (w, ϵ) in a full neighborhood of $(0, 0)$, and there $h(w, \epsilon)$ is uniformly of order ϵ. Also since

$$\frac{\partial}{\partial w}[Aw + h(w, \epsilon)] = A + \frac{\partial g}{\partial z}(z)$$

is everywhere a Hamiltonian matrix, the differential system $(*)$ is Hamiltonian for each fixed ϵ near 0.

Since $dH^{\#}$ has a generic elliptic critical point at Q_0, the eigenvalues of A are the pure imaginaries $\lambda_1, \ldots, \lambda_n$ (and $-\lambda_1, \ldots, -\lambda_n$) and are linearly independent over the rationals. In particular write $\lambda_1 = i\omega$ with real frequency $\omega \neq 0$. Thus, for $\epsilon = 0$, the differential system $(*)$ is linear with a $2\pi/\omega$-periodic solution $w = \varphi(t) = e^{At}w_0$, for some numerical unit vector w_0. Because $(*)$ is linear, for $\epsilon = 0$, the variational equation about $w = \varphi(t)$ is just the same differential system. Thus the characteristic multipliers of $\varphi(t)$ are the eigenvalues of $\exp[(2\pi/\omega)A]$, namely; 1, 1, $\exp(\pm 2\pi\lambda_2/\omega), \ldots, \exp(\pm 2\pi\lambda_n/\omega)$.

Because $\lambda_j/\lambda_1 = \lambda_j/i\omega$ for $j = 2, \ldots, n$ is never an integer, the standard perturbation theory for Hamiltonian systems guarantees the existence of a parametrized family $w = \varphi(t, \epsilon)$ of periodic solutions of the nonlinear differential system $(*)$ for all small $|\epsilon|$, with $\varphi(t, 0) = \varphi(t)$. Furthermore $\varphi(t, \epsilon)$ has period $T(\epsilon) = (2\pi/\omega) + O(\epsilon)$, and its characteristic multipliers are 1, 1, and $\exp(\pm 2\pi\lambda_j/\omega) + O(\epsilon)$ for $j = 2, \ldots, n$.

Recalling the substitution $z = \epsilon w$, we conclude that for each small $\epsilon > 0$ there is a $T(\epsilon)$-periodic orbit $z = \epsilon\varphi(t, \epsilon)$ of the original Hamiltonian system $dH^{\#}$. Moreover, for all sufficiently small $\epsilon > 0$, the initial point $z(0) = \epsilon\varphi(0, \epsilon)$ is very near to $\epsilon\varphi(0) = \epsilon w_0$ and the period $T(\epsilon)$ is very near to $2\pi/\omega$. Thus $z = \epsilon\varphi(t, \epsilon)$ must lie in the prescribed neighborhood N of the critical point Q_0. Finally, the characteristic multipliers of the periodic orbit $z = \epsilon\varphi(t, \epsilon)$ must approximate $\exp(\pm 2\pi\lambda_j/\omega)$ as $\epsilon \to 0$, and thus the characteristic frequencies must approximate the distinct irrational numbers $(\pm\lambda_j/i\omega)$ for $j = 2, \ldots, n$. Upon rearranging the list of the characteristic frequencies of $\epsilon\varphi(t, \epsilon)$, we can declare these to be $0 < \omega_2, \ldots, \omega_n < 1/2 \pmod 1$ for an appropriate choice of $\epsilon > 0$. □

Theorem 6 asserts the existence of an initial elliptic orbit γ_0 of the Hamiltonian system $dH^{\#}$. The next existence Theorem 7 produces another elliptic orbit γ_1 that encircles b_0-times around a tubular neighborhood Π centered on γ_0. To clarify this geometric description we remark that the tubular neighborhood Π around γ_0 is a topological product of an open ball B^{2n-1}, and a circle S^1 (note that Π is an orientable $2n$-manifold since it is symplectic). Moreover we demand that the compact closure $\overline{\Pi}$ is topologically $\overline{B^{2n-1}} \times S^1$ and that it forms a collar around γ_0 in the standard manner. The closed curve γ_0, restricted to a single least-period, is a generator of the fundamental group of this tube Π. We say that a periodic orbit γ_1 in Π encircles Π b_0-times, for some integer $b_0 \geq 1$, in case the closed curve γ_1, during a single period, is freely homotopic in Π to b_0-multiples of the curve γ_0.

The basic existence argument will then be repeated in an inductive proof to show that $dH^{\#}$ has an infinite sequence of elliptic orbits $\gamma_0, \gamma_1, \gamma_2, \gamma_3, \ldots$, each encircling a tubular neighborhood of the preceding orbit, with corresponding encircling multiplicities $b_0, b_1, b_2, \ldots$ for $b_j \geq 2$. Following this construction we shall then show that the sequence $\gamma_0, \gamma_1, \gamma_2, \gamma_3, \ldots$ converges to a solenoidal minimal set Σ_b defined topologically by the parameter $b = (b_0, b_1, b_2, b_3, \ldots)$.

As a further complication we shall require the existence of a prescribed minimal solenoid Σ_a with $a = (a_0, a_1, a_2, a_3, \ldots)$ for $a_j \geq 2$. Thus we must investigate the number-theoretic inter-relations of a and b corresponding to homeomorphic solenoids Σ_a and Σ_b. Hence we are led to an arithmetic-topologic degression on solenoids before advancing to the analysis of the basic existence Theorem 7.

We recall that Σ_b is topologically homeomorphic to Σ_a under the following condition: each prime-power that divides any product $(a_0 a_1 a_2 \cdots a_k)$ also divides some product $(b_0 b_1 b_2 \cdots b_l)$, and vice versa. In our algorithm for constructing Σ_b, given Σ_a, we shall choose b_0 to be a product of a prime factor q_0 of a_0, and a second factor Q_0 that is composed of further primes found in some product $(1/q_0)(a_0 \cdot a_1 \cdot a_2 \cdots a_k)$. In this way we force any prime-power factor of b_0 to divide $(a_0 \cdot a_1 \cdot a_2 \cdots a_k)$. Later choices of $b_1, b_2, \ldots$ incorporate the prime factors arising in the integers of $a = (a_0, a_1, a_2, \ldots)$ in such a way to meet the criterion: each prime-power that divides any $(b_0 \cdot b_1 \cdot b_2 \cdots b_l)$ also divides some $(a_0 \cdot a_1 \cdot a_2 \cdots a_k)$. But we must further take care in our process to pick-up all the primes, and their powers, that do arise in any such product $(a_0 \cdot a_1 \cdot a_2 \cdots a_k)$.

In order to describe our algorithm for constructing Σ_b, as a topological image of Σ_a, we begin by analyzing the prime factors ≥ 2 of the integers $a_0, a_1, a_2, \ldots$.

Case 1. Only finitely many distinct primes occur among the factors of $a_0, a_1, a_2, \ldots$.

In this case some prime $s_0 \geq 2$ has arbitrarily high powers $s_0{}^c$ that divide some finite products $(a_0 \cdot a_1 \cdot a_2 \cdots a_k)$. For definiteness take $s_0 \geq 2$ as the smallest such prime. Then s_0 must divide infinitely many of the integers of the sequence $a_0, a_1, a_2, \ldots$.

We next list the prime factors, with repetitions, for a_0, say by increasing magnitude. Thereafter list the prime factors, with repetitions, of a_1. Continue in this way to make an ordered list of the prime factors arising in the sequence $a_0, a_1, a_2, \ldots$. From this ordered list delete the prime s_0 and all its repetitions. Designate the remaining ordered list as the *usable prime list* of $a = (a_0, a_1, a_2, \ldots)$.

Case 2. There are infinitely many distinct primes among the factors of $a_0, a_1, a_2, \ldots$.

In this case make the same ordered list of all primes, with repetitions, arising in the sequence $a_0, a_1, a_2, \ldots$. Designate the total list here as the *usable prime list* of $a = (a_0, a_1, a_2, \ldots)$.

Algorithm. Let there be given a sequence of integers $a = (a_0, a_1, a_2, \ldots)$ with each $a_j \geq 2$. We shall describe a rule of choosing integers $b_j \geq 2$, as limited by the usable prime list of a.

Case 1. Take $b_0 = q_0 s_0^{c_0}$ where q_0 is the first usable prime (so $q_0 \neq s_0$; but if none such exists then take $q_0 = s_0$). The integer $c_0 \geq 1$ can be selected arbitrarily, in particular so that $s_0^{c_0} \geq s$ for any preassigned integer s. Now strike out q_0 (but not its repetitions) from the list of usable primes.

Next take $b_1 = q_1 s_0^{c_1}$ where q_1 is the first remaining usable prime (if none such exists then take $q_1 = s_0$). Again the integer $c_1 \geq 1$ is arbitrary. Now strike out q_1 from the remaining list of usable primes.

At the ℓ-th step, take $b_\ell = q_\ell s_0^{c_\ell}$ where q_ℓ is the first remaining usable prime, and $c_\ell \geq 1$ is arbitrary. This completes the description of the algorithm for selecting $b = (b_0, b_1, b_2, \ldots)$ in Case 1.

Case 2. Take $b_0 = q_0 s_1$ where q_0 is the first usable prime, and s_1 is any later prime on the list. We allow $s_1 = q_0$, but often we might demand instead that $s_1 \geq s$ for some pre-assigned large integer s. Now strike out q_0 and s_1 (but not repetitions) from the list of usable primes.

Next take $b_1 = q_1 s_2$ for q_1 the first remaining usable prime, and s_2 a usable prime later on the list. Again the choice of s_2 from the list is largely arbitrary. Now strike out q_1 and s_2 from the remaining list of usable primes.

At the ℓ-th step take $b_\ell = q_\ell s_{\ell+1}$, as before, with q_ℓ the first remaining usable prime and $s_{\ell+1}$ from later in the list of the remaining usable primes. This completes the description of the algorithm in Case 2.

LEMMA 1. *Consider a solenoid Σ_a for the parameter $a = (a_0, a_1, a_2, \ldots)$ with all $a_j \geq 2$. Let $b = (b_0, b_1, b_2, \ldots)$ be any sequence of integers selected in accord with the above algorithm. Then Σ_b is homeomorphic to Σ_a.*

Proof. If we fix attention on a finite product $(a_0 \cdot a_1 \cdot a_2 \cdots a_k)$, then all of its prime factors with appropriate repetitions will eventually enter into some finite product $(b_0 \cdot b_1 \cdot b_2 \cdots b_\ell)$. This follows since, at each stage b_j, we incorporate another prime leading the list of usable primes of a—and also at least one power of s_0 in Case 1.

On the other hand, no prime-power enters $(b_0 \cdot b_1 \cdots b_\ell)$ that does not already arise in some finite product $(a_0 \cdot a_1 \cdots a_k)$. This is a consequence of the step in the algorithm referring to the striking-out of used primes.

Thus each prime-power that divides any $(a_0 \cdot a_1 \cdots a_k)$ also divides some finite product $(b_0 \cdot b_1 \cdots b_\ell)$, and vice versa. Hence Σ_b and Σ_a are homeomorphic solenoids. □

In our construction of the elliptic orbit γ_1 which encircles a tubular neighborhood Π of the orbit γ_0, we shall require that γ_0 has a characteristic frequency ω_2 that is rational. Suppose we wish to assure that γ_1 has a least period accounting for q-encirclings, for a prescribed integer $q \geq 2$. In this situation we shall require that $\omega_2 = p/q$, where p and q are relatively prime so $(p, q) = 1$. But our geometric perturbation methods in Theorem 7 will allow us to adjust the rational $\omega_2 = p/q$ only within a narrow real interval L. Accordingly the next number-theoretic lemma will be useful.

LEMMA 2. *Let L be a compact interval in $(0, \infty)$, and let $q_0 \geq 2$ be a prime. Then there exists an integer s such that:*

for each prime-power $s_0{}^c \geq s$ there exists a rational number $p/q \in L$ with $(p, q) = 1$ and $q = q_0 s_0{}^c$.

Proof. Take $s = \max\{2, 6/|L| \cdot q_0\}$ where $|L|$ is the positive length of the interval L. Let $s_0{}^c \geq s$ so $q = q_0 s_0{}^c \geq q_0 s \geq 6/|L|$. Then $1/q \leq |L|/6$ so there exists a smallest odd integer $p_0 \geq 1$ for which $p_0/q \in L$.

Several cases give rise to various definitions of the integer p so $(p, q) = 1$.

If $q_0 = 2$ and $s_0 = 2$, then q is a power of 2 and we take $p = p_0$.

Now assume $q_0 = 2$ and $s_0 \geq 3$. In this case take $p = p_0$ unless s_0 divides p_0 in which case take $p = p_0 + 2$. The case $q_0 \geq 3$ and $s_0 = 2$ leads again to a choice between p_0 and $p_0 + 2$.

Finally assume $q_0 \geq 3$ and $s_0 \geq 3$. Take $p = p_0$ if $(p_0, q_0) = (p_0, s_0) = 1$. Otherwise take $p = p_0 + 1$ or $p = p_0 + 2$ whichever yields the relatively prime demands $(p, q_0) = 1$ and $(p, s_0) = 1$.

In every case we augment p_0 by at most 2 to obtain p. Since $(p_0 - 2)/q$ is a lower bound for L, and since $1/q < |L|/6$, we compute $(p_0 + 2)/q = (p_0/q) + (2/q) \in L$. Thus, in every case, we have defined p so $(p, q) = 1$ with $p/q \in L$. □

Now we turn to the existence theorem that lies at the heart of our technical analysis.

THEOREM 7. *Let $dH^\#$ be a Hamiltonian system in the generic set $\mathfrak{S}_1 \cap \mathfrak{S}_2 \cap \mathfrak{S}_3 \subset \mathfrak{H}^k$, for $4 \leq k \leq \infty$, on a symplectic 2n-manifold M, for $2n \geq 4$. Let γ_0 be an elliptic periodic orbit of $dH^\#$, having distinct characteristic frequencies. Let Π be a tubular neighborhood of γ_0 in M and let $q_0 \geq 2$ be a given prime.*

Then there exists an integer $s \geq 2$ such that: for each prime-power $s_0{}^c \geq s$ there exists a periodic orbit γ_1 of $dH^\#$ in the tube Π, with the two properties,

i) *γ_1 encircles the tube Π exactly $q = q_0 s_0{}^c$ times before closing after its least period,*
and

ii) *γ_1 is an elliptic orbit with distinct characteristic frequencies.*

Proof. The elliptic periodic orbit γ_0 of $dH^\#$ has distinct characteristic frequencies, say $0 < \omega_2, \ldots, \omega_n < 1/2$. Thus γ_0 is nondegenerate and it is embedded in a local 2-band of elliptic periodic orbits $\gamma(h)$, as parameterized by the energy level h of the flow of $dH^\#$. Moreover, the corresponding characteristic frequencies $0 < \omega_2(h), \ldots, \omega_n(h) < 1/2$ are distinct for $\gamma(h)$, at least for small energy levels h where we normalize the energy by $\gamma(0) = \gamma_0$. Furthermore all these elliptic orbits $\gamma(h)$ lie inside the tube Π centered on γ_0, provided we suitably restrict the energy level of h.

We shall break the proof down into several steps. The first step deals with the restriction of the energy levels of h, and the reselection of the initial orbit γ_0 in $\gamma(h)$ so that $\omega_2 = p/q$, is rational with $q = q_0 s_0{}^c$. The remaining steps then exploit the rational frequency $\omega_2 =$

p/q, through fixed-point methods, to produce the required elliptic orbit γ_1 that encircles the tube Π q-times.

i) *Adjustment of the initial elliptic orbit* γ_0.

By the definition and generic property of the set $\mathfrak{S}_2$, as established in Theorem 4, we can assume that some $\omega_j(h)$ is non-constant. For definiteness assume that $d\omega_2/dh > 0$ on some subinterval (h_0, h_1), within the 2-band of the periodic orbits $\gamma(h)$ that lie within the tube Π. In addition, from the definition of the generic set $\mathfrak{S}_3$, in accord with Theorem 5, we can assume that the twist coefficient $Tw(c_{ij}(h))$ vanishes nowhere on some open subinterval $(h_0', h_1') \subset (h_0, h_1)$.

Next we seek a value $\tilde{h} \in (h_0', h_1')$ where $\omega_2(\tilde{h}) = p/q$ for $q = q_0 s_0^c$ with $(p, q) = 1$, for the chosen prime-power $s_0^c \geq s$. The integer $s \geq 2$ to be specified first.

Take a compact interval L in the range of $\omega_2(h)$, for $h_0' < h < h_1'$ where $d\omega_2/dh > 0$. By the above Lemma 2, given L and q_0 there exists the appropriate integer $s \geq 2$ and thereafter we take any prime-power $s_0^c \geq s$. Under these conditions there exists a rational number $p/q \in L$ with $(p, q) = 1$ and $q = q_0 s_0^c$. Hence there exists the energy value $\tilde{h} \in (h_0', h_1')$ for which $\omega_2(\tilde{h}) = p/q$.

Now we rename the energy levels, that is translate the energy range, and reselect the initial orbit to be $\gamma(\tilde{h})$ which we call γ_0 and thus re-adjust the energy level so $\tilde{h} = 0$. That is, we keep the original notation $\gamma_0 = \gamma(0)$ and use γ_0 as the initial elliptic orbit lying within the 2-band $\gamma(h)$ of elliptic orbits of $dH^{\#}$, all of which are inside the tube Π for $-\hat{h} < h < \hat{h}$. We mention that each $\gamma(h)$ encircles the tube Π just once, since the period of $\gamma(h)$ varies continuously with h, and so Π still forms the standard collar around γ_0.

Therefore, after this re-selection of γ_0 upon an arbitrarily slight shift within the given 2-band $\gamma(h)$ of elliptic orbits, we can assume:

α) the characteristic frequencies are everywhere distinct,

$$0 < \omega_2(h), \ldots, \omega_n(h) < 1/2 \qquad \text{on } -\hat{h} < h < \hat{h},$$

and

β) $d\omega_2/dh > 0$ and $Tw(c_{ij}(h)) \neq 0$ on $-\hat{h} < h < \hat{h}$, and at the level $h = 0$ where $\gamma(0) = \gamma_0$,

γ) $\omega_2(0) = p/q$ with $(p, q) = 1$ and $q = q_0 s_0{}^c$ for some large prime-power $s_0{}^c \geq s$, (automatically $q \geq 6$ since the Birkhoff form is valid in $\mathfrak{S}_3$) and, by Theorem 3,

δ) $\omega_3(0), \omega_4(0), \ldots, \omega_n(0)$ are all irrational.

ii) *Conditions for fixed-points of the iterated Poincaré map around* γ_0.

We shall proceed to find a q-periodic orbit γ_1 of $dH^{\#}$, q-encircling the tubular neighborhood Π centered on γ_0. To accomplish this goal we shall seek a fixed-point of the q-th iterated $P^{[q]}$ of the Poincaré map P defined by the flow of $dH^{\#}$ around the given periodic orbit γ_0. That is, take a transversal $(2n - 1)$-section Σ to the orbit γ_0 at some point $Q = \gamma_0 \cap \Sigma$ and define P as the map of first-return, along the trajectories of $dH^{\#}$, of a neighborhood of Q in Σ back into Σ. Then $P^{[q]}$ describes the q-th return to Σ, and we seek a nontrivial fixed-point of the map $P^{[q]}$.

Take a local chart of canonical coordinates $(x^1, x^2, \ldots, x^n, y_1, y_2, \ldots, y_n)$ centered at Q in M, wherein $H = y_1$ and Σ is the hyperplane $x^1 = 0$. Then $(x^2, \ldots, x^n, y_2, \ldots, y_n)$ are symplectic coordinates for each slice of Σ defined by an energy level $y_1 = h$. Furthermore we can require that the 2-band of orbits $\gamma(h)$ meets Σ in the locus $x^j = 0$, $y_j = 0$ for $j = 2, \ldots, n$, as h varies on Σ. In addition (x^j, y_j) can be taken to be Birkhoff normal coordinates that display the Poincaré map P_h, depending smoothly on the energy level h, in the Birkhoff normal form in accord with the properties of $\mathfrak{S}_3$ and the Birkhoff Theorem in Section 3 above.

The Poincaré map P around γ_0 is now displayed, on each energy level h of Σ, by the symplectic map:

$$P_h : (x^j, y_j) \to (X^j(x, y, h), Y_j(x, y, h)),$$

in terms of the parameter-symplectic coordinates $(x^2, \ldots, x^n, y_2, \ldots, y_n)$ and h. In this format we can examine P_h and its r-th iterate $P_h{}^{[r]}$. Here we write

$$P_h{}^{[r]} : (x^j, y_j) \to (X^{j[r]}(x, y, h), Y_j{}^{[r]}(x, y, h))$$

and we shall emphasize the linear terms which are rotations in these Birkhoff normal coordinates (x^j, y_j).

The search for periodic orbits of $dH^{\#}$ that encircle γ_0 r-times leads to the condition specified by the fixed-point equations $P_h{}^{[r]} = Id$. We can display these fixed-point equations as

$$\begin{aligned}
x^2 \cos 2\pi r\omega_2 + y_2 \sin 2\pi r\omega_2 + \cdots &= x^2 \\
-x^2 \sin 2\pi r\omega_2 + y_2 \cos 2\pi r\omega_2 + \cdots &= y_2 \\
x^3 \cos 2\pi r\omega_3 + y_3 \sin 2\pi r\omega_3 + \cdots &= x^3 \\
-x^3 \sin 2\pi r\omega_3 + y_3 \cos 2\pi r\omega_3 + \cdots &= y_3 \\
&\vdots \\
x^n \cos 2\pi r\omega_n + y_n \sin 2\pi r\omega_n + \cdots &= x^n \\
-x^n \sin 2\pi r\omega_n + y_n \cos 2\pi r\omega_n + \cdots &= y_n .
\end{aligned} \tag{F.P.E.}$$

Here $\omega_j = \omega_j(h)$ but $\omega_2(0) = p/q$ and furthermore $\omega_3(0), \ldots, \omega_n(0)$ are distinct irrationals. The omitted remainder terms are of order $O(\Sigma^n{}_{j=2} |x^j|^2 + |y_j|^2)$ near $x^j = 0$, $y_j = 0$, uniformly in h.

These maps and fixed-point conditions can be analysed more easily in symplectic polar coordinates

$$u_j = \frac{(x^j)^2 + (y_j)^2}{2}, \qquad \theta_j = \text{arc tan } y_j/x_j \text{ for } j = 2, \ldots, n,$$

as are introduced for the Birkhoff normal form. In these coordinates we can write the Poincaré map

$$\begin{aligned}
P_h : u_j &\to U_j = u_j + \cdots \\
\theta_j &\to \Theta_j = \theta_j + 2\pi\omega_j + \sum_{l=2}^{n} c_{jl} u_l + \cdots \pmod{2\pi}.
\end{aligned}$$

Then the fixed-point equations $P_h{}^{[r]} = Id$ become

$$\begin{aligned}
\text{(F.P.E.)} \quad u_j &\to U_j{}^{[r]} = u_j + \cdots = u_j \\
\theta_j &\to \Theta_j{}^{[r]} = \theta_j + 2\pi r\omega_j + r \sum_{l=2}^{n} c_{jl} u_l + \cdots = \theta_j \pmod{2\pi}.
\end{aligned}$$

Here $\omega_j(h)$ are the characteristic frequencies of $\gamma(h)$ and $c_{jl}(h)$ is the twist matrix, depending smoothly on h. The omitted terms are of order $O(\|u\|^2)$ near $u = 0$, uniformly in θ and h.

iii) *Absence of short periodic orbits encircling* γ_0.

Fix any integer $1 \leq r < q$ and seek solutions of the fixed-point equations $P_h{}^{[r]} = Id$. Of course $x^j(h) = 0$, $y_j(h) = 0$ yields a trivial solution corresponding to the periodic orbit $\gamma(h)$, which has an encircling multiplicity of just $+1$ around γ_0. We must show that there are no other nontrivial solutions of (F.P.E.), provided the tube Π is suitably narrowed with restrictions on $|h| < \hat{h}$ whenever necessary. We usually assume that such narrowing of Π, and restrictions of $\hat{h}$ are done without any special mention or notational change.

Examine (F.P.E.) in the symplectic cartesian coordinates (x^j, y_j), especially with regard to the linear terms. Note that the $(n - 1) \times (n - 1)$ determinant of the linearized equations, at $h = 0$, is just the product of $(n - 1)$ determinants of size 2×2, namely,

$$\det \begin{vmatrix} \cos 2\pi rp/q - 1 & \sin 2\pi rp/q \\ -\sin 2\pi rp/q & \cos 2\pi rp/q - 1 \end{vmatrix}$$

and

$$\det \begin{vmatrix} \cos 2\pi r\omega_3 - 1 & \sin 2\pi r\omega_3 \\ -\sin 2\pi r\omega_3 & \cos 2\pi r\omega_3 - 1 \end{vmatrix},$$

$$\ldots, \det \begin{vmatrix} \cos 2\pi r\omega_n - 1 & \sin 2\pi r\omega_n \\ -\sin 2\pi r\omega_n & \cos 2\pi r\omega_n - 1 \end{vmatrix}.$$

Since $\omega_3(0), \ldots, \omega_n(0)$ are all irrational, all of the last $(n - 2)$ of these determinants are different from zero. Also note that rp/q, with $(p, q) = 1$ and $r < q$, is not an integer and so the first of these determinants is different from zero. Therefore the determinant of the linear terms of the (F.P.E.), $P_h{}^{[r]} - Id = 0$, must be nonzero at $h = 0$ and also for all small $|h|$.

Under these conditions the implicit function theorem guarantees that only the trivial solution $x^j = y_j = 0$ exists, provided $|x^j| + |y_j| + |h|$ is suitably small. Hence, within a suitably restricted tubular neighborhood Π about γ_0, there are no periodic orbits of $dH^{\#}$ with r-encircling multiplicities for $1 \leq r < q$; excepting the known orbits $\gamma(h)$ of the standard 2-band.

iv) *Existence of q-encircling orbits: reduction to 2-surface $S(h)$.*

If we set $r = q$ then the first two equations of the system $P_0^{[q]} = Id$ reduce to

$$x^2 + \cdots = x^2$$

$$y_2 + \cdots = y_2.$$

Since the 2×2 determinant of the linear terms is zero, the implicit function theorem is not directly applicable to the full system of $(2n - 2)$-equations. However the last $(2n - 4)$ of the equations of the system $P_h^{[q]} = Id$ have a corresponding $2 \times (n - 2)$ determinant that is the product

$$\det \begin{vmatrix} \cos 2\pi q\omega_3 - 1 & \sin 2\pi q\omega_3 \\ -\sin 2\pi q\omega_3 & \cos 2\pi q\omega_3 - 1 \end{vmatrix}$$

$$\cdots \det \begin{vmatrix} \cos 2\pi q\omega_n - 1 & \sin 2\pi q\omega_n \\ -\sin 2\pi q\omega_n & \cos 2\pi q\omega_n - 1 \end{vmatrix} \neq 0.$$

This implies that we can solve the last $(2n - 4)$-equations of the (F.P.E.) for $(x^3, \ldots, x^n, y_3, \ldots, y_n)$ in terms of the variables (x^2, y_2) near $(0, 0)$, with the parameter h also near 0. That is, there exist

$$x^\ell = \sigma^\ell(x^2, y_2, h), \qquad y_l = \tau_\ell(x^2, y_2, h) \quad \text{for} \quad \ell = 3, \ldots, n$$

where the C^k-functions $\sigma^l(x^2, y_2, h)$ and $\tau_l(x^2, y_2, h)$ satisfy the conditions

$$\sigma^\ell(0, 0, h) = 0, \qquad \tau_\ell(0, 0, h) = 0$$

and

$$\frac{\partial \sigma^\ell}{\partial x^2} = \frac{\partial \sigma^\ell}{\partial y_2} = 0, \qquad \frac{\partial \tau_\ell}{\partial x^2} = \frac{\partial \tau_\ell}{\partial y_2} = 0 \quad \text{at} \quad (0, 0, h).$$

Therefore, for each small $|h|$, we can define a 2-surface $S(h)$,

$$S(h): x^{\ell} = \sigma^{\ell}(x^2, y_2, h), \qquad y_{\ell} = \tau_{\ell}(x^2, y_2, h) \quad \text{for} \quad \ell = 3, \ldots, n.$$

For each h the 2-surface $S(h)$ is tangent to the (x^2, y_2)-plane at the origin, and lies within the h-energy slice of the section Σ. On $S(h)$ the coordinates $(x^3, \ldots, x^n, y_3, \ldots, y_n)$ vary as (x^2, y_2) varies over a neighborhood of the origin in the (x^2, y_2)-plane. In this sense we coordinatize each $S(h)$ by (x^2, y_2), and compute its position in the transversal Σ by $x^{\ell} = \sigma^{\ell}(x^2, y_2, h)$, $y_{\ell} = \tau_{\ell}(x^2, y_2, h)$, for each h.

The significance of the surface $S(h)$ is that each point (x^2, y_2) on $S(h)$, whose remaining coordinates are $(\sigma^{\ell}, \tau_{\ell})$ and h in Σ, is mapped by $P_h{}^{[q]}$ to an image point in Σ whose coordinates $x^{\ell} = \sigma^{\ell}$, $y_{\ell} = \tau_{\ell}$ and h are unchanged. In other words, for each point $(x^2, \sigma^{\ell}, y_2, \tau_{\ell}, h)$ on $S(h)$, only the (x^2, y_2) coordinates are changed by the map $P_h{}^{[q]}$, and the remaining coordinates $(\sigma^{\ell}, \tau_{\ell}, h)$ are unchanged. Thus the problem of finding a fixed-point of $P_h{}^{[q]}$, within the transversal Σ of the tube Π around γ_0, reduces to finding a point with fixed coordinates $(\hat{x}^2, \hat{y}_2)$ on the surface $S(h)$.

v). *Curve $\mathfrak{C}(h)$ of angular invariance on $S(h)$.*

We use the coordinates (x^2, y_2) to specify points on the surface $S(h)$, and further simplify the calculations by introducing the symplectic polar coordinates on this surface, according to:

$$u = \frac{(x^2)^2 + (y_2)^2}{2}, \qquad \theta = \arctan y_2/x_2.$$

Then the q-th iterate of the Poincaré map $P_h{}^{[q]}$, when restricted to $S(h)$, can be described by the first pair of the (F.P.E.),

$$u \to U^{[q]} = u + \cdots = u$$

$$\theta \to \Theta^{[q]} = \theta + 2\pi q \omega_2(h) + q c_{22} u + q \sum_{\ell=3}^{n} c_{2\ell} U_{\ell} + \cdots = \theta.$$

Here we have written (u, θ) for (u_2, θ_2), so that we can define the surface $S(h)$ by

$$S(h): u_{\ell} = u_{\ell}(u, \theta, h), \qquad \theta_{\ell} = \theta_{\ell}(u, \theta, h) \quad \text{for} \quad \ell = 3, \ldots, n.$$

Since the functions

$$u_{l'}(u, \theta, h) = \frac{(\sigma^{l'})^2 + (\tau_{l'})^2}{2} = O(|x^2|^4 + |y_2|^4) = O(u^2),$$

we can write the pair of (F.P.E.) in the form

$$u = u + \cdots$$

$$\theta = \theta + 2\pi q\left[\frac{p}{q} + \frac{d\omega_2}{dh}(0)h\right] + qc_{22}(0)u + \cdots,$$

where the error is of order $O(u^2 + h^2)$.

Now write $\alpha = 2\pi q(d\omega_2/dh)(0)$ and $\beta = -qc_{22}(0)$ and then we have our basic equations for a fixed point of $P_h{}^{[q]}$ on $S(h)$:

$$u = u + \cdots$$

$$\theta = \theta + \alpha h - \beta u + \cdots \qquad (\text{mod } 2\pi).$$

We know that $\alpha \neq 0$, from our selection of the initial elliptic orbit γ_0 of $dH^\# \in \mathfrak{S}_2$, as explained earlier. The condition $\beta \neq 0$ is a consequence of the nonvanishing twist of the orbit γ_0 of $dH^\# \in \mathfrak{S}_3$. In fact, this calculation is the motivation of the twist coefficient $Tw(c_{jl}) = c_{22} \cdot c_{33} \cdots c_{nn}$.

We assume, for convenience, that both α and β are positive constants in order to compare the consequences of a positive shift in the energy h versus a positive radial dilation in u. If α and β were of opposite signs, we could shift h negatively while dilating u positively to obtain similar conclusions. Thus we take $\alpha > 0$ and $\beta > 0$.

Next define a curve $\mathfrak{C}(h)$ on the surface $S(h)$,

$$\mathfrak{C}(h): u = u(\theta, h), \qquad \text{for } 0 \leq \theta < 2\pi,$$

by the solution of the angular (F.P.E.)

$$0 = \alpha h - \beta u + \cdots.$$

Since the error term is $O(u^2 + h^2)$, uniformly in θ, the implicit function theorem enables us to solve for u according to

$$u = u(\theta, h) = \frac{\alpha}{\beta} h + \cdots$$

where the error is $O(h^2)$, uniformly in θ. Since $u(\theta, h)$ is positive for all small $h > 0$, this locus defines a smooth simple closed curve $\mathfrak{C}(h)$ encircling the origin in the surface $S(h)$.

For each small $h > 0$ the closed curve $\mathfrak{C}(h)$ has the important property that each point $(u(\hat{\theta}, h), \hat{\theta}) \in \mathfrak{C}(h)$ has its angular coordinate $\hat{\theta}$ preserved under the map $P_h{}^{[q]}$. In other words, such a point on $\mathfrak{C}(h)$ would be the desired fixed point of $P_h{}^{[q]}$ if its radial coordinate $\hat{u} = u(\hat{\theta}, h)$ satisfied the remaining radial (F.P.E.)

$$u(\hat{\theta}, h) = u(\hat{\theta}, h) + \cdots.$$

As is evident from the format of this last radial equation, the process of solution for $\hat{\theta}$ must be rather delicate since the usual implicit function theorem is not applicable here. In order to motivate our proof we remark that in the special case where $S(h)$ is an invariant symplectic linear space, say $x^l = 0$, $y_l = 0$ for $l = 3, \ldots, n$, then the area-preservation property of $P_h{}^{[q]}$ would show that the closed curve $\mathfrak{C}(h)$ must meet its image in this plane. In such a situation any point of intersection of $\mathfrak{C}(h)$ with its image would necessarily be a fixed point of the map $P_h{}^{[q]}$. But $S(h)$ is not necessarily a linear plane, and so we must use a more complicated argument based on other invariants of symplectic geometry (see [3] for a similar argument).

vi) *Existence of q-encircling orbits: fixed points on the curve* $\mathfrak{C}(h)$.

Consider the (F.P.E.) on the surface $S(h)$

$$u = u + \cdots$$

$$\theta = \theta + \alpha h - \beta u + \cdots \qquad (\text{mod } 2\pi)$$

for constants $\alpha > 0$, $\beta > 0$ with error $O(u^2 + h^2)$. For all suitably small $|h|$ we obtain the unique solution of the angular equation

$$u = u(\theta, h) = \frac{\alpha}{\beta} h + O(h^2),$$

compatible with the restrictions on the magnitude of u imposed by the tube Π. For $h \leq 0$ there are no such points on $S(h)$ with $u > 0$; hence no such geometric point exists. Thus for each $h \leq 0$ the only candidates for q-encircling periodic orbits of $dH^{\#}$ in the tube Π about γ_0 are just the 1-encircling orbits $\gamma(h)$ of the 2-band through $\gamma(0) = \gamma_0$. Hence we need only examine the locus for $h > 0$ which defines the closed curve,

$$\mathfrak{C}(h): u = u(\theta, h), \qquad \text{for all small } h > 0.$$

Of course, the tube Π and the corresponding energy domain $-\hat{h} < h < \hat{h}$ are always assumed to be suitably restricted.

The map $P_h^{[q]}$ is symplectic on each energy level $\Sigma(h)$ in the transversal Σ to γ_0 across the tube Π. Accordingly, $P_h^{[q]}$ must preserve the symplectic 2-form $\Sigma^n_{j=2}\, du_j \wedge d\theta_j$ in both sets of canonical coordinates (u_j, θ_j) and $(U_j^{[q]}, \Theta_j^{[q]})$. That is, the identity in (u_j, θ_j) must hold,

$$d \sum_{j=2}^{n} (U_j^{[q]} d\Theta_j^{[q]} - u_j d\theta_j) = 0.$$

Within the simply-connected manifold $\Sigma(h)$ this closed 1-form is the differential dW of some real function $W(u_2, \ldots, u_n, \theta_2, \ldots, \theta_n)$, that is,

$$dW = \sum_{j=2}^{n} (U_j^{[q]} d\Theta_j^{[q]} - u_j d\theta_j).$$

We now restrict W and the corresponding equality of differential forms to the submanifold consisting of the smooth curve $\mathfrak{C}(h)$. Along $\mathfrak{C}(h)$ find the equality

$$dW = (U_2^{[q]} - u) d\theta.$$

Here the functions $U_2^{[q]}$ and $u_2 = u$, $\theta_2 = \theta$ are evaluated along the smooth curve $\mathfrak{C}(h)$.

Since $\mathfrak{C}(h)$ is a compact 1-manifold, so diffeomorphic to a circle, there must exist critical points of W on $\mathfrak{C}(h)$—say at the minimum and the maximum values of W on $\mathfrak{C}(h)$. At each such critical point $dW = 0$ so $U_2^{[q]} = u_2$ and hence this establishes the existence of at least two fixed-points of the map $P_h^{[q]}$ on each curve $\mathfrak{C}(h)$, for each small $h > 0$.

vii) *Index computations for elliptic orbits.*

For each small $h > 0$ we obtain at least one nontrivial q-encircling orbit of $dH^{\#}$, determined by a fixed-point of $P_h^{[q]}$ on $\mathfrak{C}(h)$, lying within the tube Π around γ_0. Earlier steps in the proof have shown that such q-encircling orbits cannot reduce to r-encircling orbits for any $1 \leq r < q$. We must now show that our construction yields a nondegenerate elliptic orbit q-encircling Π.

First delete from the energy interval $-\hat{h} < h < \hat{h}$ the countable set of values of h for which there exist periodic orbits of $dH^{\#}$ having repeated characteristic multipliers. This energy set D is countable since $dH^{\#} \in \mathfrak{S}_2 \subset \mathfrak{R}$, and therefore $(-\hat{h}, \hat{h}) - D$ is dense in the real interval $(-\hat{h}, \hat{h})$.

Consider any small h in the set $(0, \hat{h}) - D$. Then each of the q-encircling periodic-orbits is located by a fixed-point of $P_h^{[q]}$ on the compact set $\mathfrak{C}(h)$. Since every periodic orbit at energy level h is nondegenerate, there are only a finite number $e(h)$ of these q-encircling orbits, and we denote these by $\gamma_1{}^1(h), \gamma_1{}^2(h), \ldots, \gamma_1{}^{e(h)}(h)$.

Next we compute the local topological degree of each of these fixed-points of $P_h^{[q]}$, namely at the points $\gamma_1{}^1(h) \cap \Sigma(h), \ldots, \gamma_1{}^{e(h)}(h) \cap \Sigma(h)$, within the invariant h-level $\Sigma(h)$. This local degree, for each such fixed-point $\gamma_1{}^e(h) \cap \Sigma(h)$, is just the topological index of the vector field produced in $\Sigma(h)$ by joining each point (u_j, θ_j) to its image $(U_j{}^{[q]}, \Theta_j{}^{[q]})$. Thus we compute

$$\text{degree } \gamma_1{}^e(h) \cap \Sigma(h) = \text{sgn det } |dP_h{}^{[q]} - I|,$$

where the Jacobian matrix $dP_h{}^{[q]}$ is evaluated at the given fixed-point $\gamma_1{}^e(h) \cap \Sigma(h)$, for $e = 1, 2, \ldots, e(h)$. Since the periodic orbit $\gamma_1{}^e(h)$ is nondegenerate, $dP_h{}^{[q]}$ has no eigenvalue $+1$ so the degree depends only on the sign of the indicated determinant.

As the parameter h increases through the dense set $(0, \hat{h}) - D$, the index of the corresponding vector field in $\Sigma(h)$ can be computed over a given sphere S^{2n-3} around the origin in the $(x^2, \ldots, x^n, y_2, \ldots, y_n)$-space. The index over S^{2n-3} will be independent of the parameter h, provided all of the points $\gamma(h) \cap \Sigma(h), \gamma_1{}^1(h) \cap \Sigma(h), \ldots, \gamma_1{}^{e(h)}(h) \cap \Sigma(h)$ lie inside the sphere S^{2n-3}. This total index must be the degree of the fixed-point $\gamma_0 \cap \Sigma(0)$ corresponding to the unique fixed-point of $P_0{}^{[q]}$ at energy level $h = 0$. Moreover the trivial 1-encircling periodic orbits $\gamma(h)$ all contribute the same value as does $\gamma(0) = \gamma_0$ to this total index, and hence the extra q-encircling orbits must contribute a total of zero:

$$\sum_{e=1}^{e(h)} \text{degree } \gamma_1{}^e(h) \cap \Sigma(h) = 0.$$

Let us examine these degrees as $h \to 0+$ through the dense set $(0, \hat{h}) - D$. Note that the diameters of the closed curves $\mathfrak{C}(h)$ in $\Sigma(h)$ tend towards zero, and so all the fixed-points $\gamma_1{}^e(h) \cap \Sigma(h)$ tend towards $\gamma_0 \cap \Sigma(0)$. Hence the characteristic multipliers $\xi_2{}^e$, $\xi_3{}^e$, $\ldots$, $\xi_n{}^e$ (and their reciprocals) tend towards those of γ_0 under the q-iterated Poincaré map $P_0{}^{[q]}$. Consequently, as $h \to 0+$,

$$\xi_2{}^e = 1 + o(1)$$

$$\xi_l{}^e = e^{2\pi i q \omega_l} + o(1) \qquad \text{for } l = 3, \ldots, n.$$

Now $\omega_3(0)$, $\ldots$, $\omega_n(0)$ are distinct irrational numbers, and so $\xi_l{}^e$ are distinct complex numbers of unit modulus with complex conjugates $\overline{\xi_l{}^e} = 1/\xi_l{}^e$. Our goal is to show that $\xi_2{}^e$ is also on the complex unit circle, so that $\gamma_1{}^e(h)$ would then be elliptic.

The product $(\xi_l{}^e - 1)(1/\xi_l{}^e - 1)$, of two complex conjugates, is positive. Thus the degree of $P_h{}^{[q]}$ at $\gamma_1{}^e(h) \cap \Sigma(h)$ is

$$\text{sgn det } |dP_h{}^{[q]} - I| = \text{sgn } (\xi_2{}^e - 1)(1/\xi_2{}^e - 1).$$

Now each of the q-encircling periodic orbits $\gamma_1{}^e(h)$, for $e = 1, \ldots, e(h)$, is nondegenerate and thus the corresponding local degree is $(+1)$ or (-1). Because the sum of all these local degrees must total zero, there exists at least one of these orbits with local degree $(+1)$. For each small $h \in (0, \hat{h}) - D$ let $\gamma_1{}^1(h)$ yield a fixed-point of $P_h{}^{[q]}$ with local degree of $(+1)$.

The two cases of interest are when $\xi_2{}^1$ is real, and when $\xi_2{}^1$ is nonreal on the unit circle (since $\gamma_1{}^1(h)$ is nondegenerate, $\xi_2{}^1 \neq +1$). In the first case write $\xi_1{}^1 = 1 + \zeta$ for real $\zeta > 0$, so sgn $(\xi_1{}^1 - 1)(1/\xi_1{}^1 - 1) = \text{sgn } (\zeta)(-\zeta/1 + \zeta) = -1$. Hence the first case is ruled out.

In the second case $\xi_1{}^1 = \cos \zeta + i \sin \zeta$, so sgn $(\xi_1{}^1 - 1)$ $(\bar{\xi}_1{}^1 - 1) = \text{sgn}\,(2 - 2\cos \zeta) = +1$. Thus the second case must hold, as so $\gamma_1{}^1(h)$ is a nondegenerate elliptic orbit.

For any suitable small $h \in (0, \hat{h}) - D$ rename the nondegenerate elliptic orbit $\gamma_1 = \gamma_1{}^1(h)$. Then γ_1 is a nondegenerate elliptic orbit of $dH^\#$ and it encircles within the tube Π exactly q-times before closing. Moreover the characteristic frequencies of γ_1 are very near to the distinct numbers $1, q\omega_3(0), \ldots, q\omega_n(0)$, as required in the theorem. □

We have finally assembled all the results needed for the demonstration of our Principal Theorem, as stated in Section 1 earlier. Here we shall discuss Hamiltonian systems $dH^\#$ of class C^k in the Baire space $\mathfrak{H}^k$, for any fixed $4 \leq k \leq \infty$, on any symplectic $2n$-manifold M, for $2n \geq 4$. Our results apply to any $dH^\#$ in the generic set $\mathfrak{S}_1 \cap \mathfrak{S}_2 \cap \mathfrak{S}_3$ which is assumed to possess a nondegenerate elliptic periodic orbit γ_0. If we further assume that $dH^\# \in \mathfrak{S}_0$ and that M is compact, then Theorem 6 assures the existence of such an initial elliptic orbit γ_0 near a generic elliptic critical point. For simplicity of exposition we phrase our Principal Theorem in this latter case, and for emphasis we name the generic set

$$\mathfrak{M}_\Sigma = \mathfrak{R} \cap \mathfrak{S}_0 \cap \mathfrak{S}_1 \cap \mathfrak{S}_2 \cap \mathfrak{S}_3.$$

Principal Theorem. *Let $\mathfrak{H}^k$ be the space of Hamiltonian dynamical systems on the compact symplectic manifold M. Then there exists a generic set $\mathfrak{M}_\Sigma \subset \mathfrak{H}^k$ such that:*

for each Hamiltonian system $dH^\# \in \mathfrak{M}_\Sigma$, and for each solenoid Σ_a, there exists a minimal set for the flow of $dH^\#$ that is homeomorphic to Σ_a.

Proof.

Remark. Take a Hamiltonian C^k-vector field $dH^\#$, for $4 \leq k \leq \infty$, on any symplectic $2n$-manifold M, for $2n \geq 4$. Assume $dH^\#$ lies in the generic set $\mathfrak{S}_1 \cap \mathfrak{S}_2 \cap \mathfrak{S}_3$ and has a nondegenerate elliptic orbit γ_0—as would be guaranteed by the hypotheses $dH^\# \in \mathfrak{M}_\Sigma$ and M compact. Let Π be a tubular neighborhood of γ_0 and let Σ_a for parameter $a = (a_0, a_1, a_2, a_3, \ldots)$ with $a_j \geq 2$ be a given solenoid. Then our proof will establish the existence of a minimal set for $dH^\#$ in Π, which is homeomorphic to Σ_a. We break the proof of the theorem into several steps.

i). *Existence of the initial elliptic orbit* γ_0.

The Hamiltonian function H for the vector field $dH^\#$ must assume its minimum value at some critical point Q_0 on the compact manifold M. Since $dH^\# \in \mathfrak{S}_0$, the critical point Q_0 must be a generic elliptic critical point. By Theorem 6 there exists a nondegenerate elliptic orbit γ_0 of $dH^\#$ near Q_0, and all the characteristic frequencies $0 < \omega_2{}^0, \ldots, \omega_n{}^0 < 1/2$ of γ_0 are distinct.

ii). *Construction of the first encircling orbit* γ_1.

Take a tubular neighborhood Π_0 of γ_0. In the parameter $a = (a_0, a_1, a_2, a_3, \ldots)$ let $q_0 \geq 2$ be the first prime in the usable prime list of a, in the sense of the arithmetic analysis and the algorithm preceding Theorem 7.

According to Theorem 7 there exists an integer $s \geq 2$ such that, for each prime-power $s_0{}^{c_0} \geq s$, there is a b_0-encircling periodic orbit γ_1 of $dH^\#$ in the tube Π_0, where $b_0 = q_0 s_0{}^{c_0}$ (in Case 1 of the algorithm, or $b_0 = q_0 s_1$ in Case 2) and $s_0{}^{c_0} \geq s$ (or $s_1 \geq s$) is chosen in accord with the above arithmetic algorithm. Moreover γ_1 is a nondegenerate elliptic orbit with distinct characteristic frequencies $0 < \omega_2{}^1, \ldots, \omega_n{}^1 < 1/2$.

Finally take a narrow tubular neighborhood Π_1 of γ_1, with the closure $\bar{\Pi}_1$ lying inside Π_0 and encircling it b_0-times. Now the procedure can be repeated to determine a second elliptic orbit γ_2 that encircles Π_1 b_1-times; and hence γ_2 encircles $\Pi_0 (b_0 b_1)$-times.

iii). *Inductive definition for the sequence of orbits* $\gamma_0, \gamma_1, \gamma_2, \gamma_3, \ldots$.

Suppose that the periodic orbits $\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_\ell$ of $dH^\#$ have already been determined in the fashion indicated in the preceding step. In particular, each of $\gamma_0, \gamma_1, \ldots, \gamma_\ell$ is a nondegenerate elliptic orbit of $dH^\#$, with distinct characteristic frequencies $0 < \omega_2{}^g, \ldots, \omega_n{}^g < 1/2$ for $0 \leq g \leq \ell$.

Moreover suppose that there are specified tubular neighborhoods Π_0 of γ_0, Π_1 of $\gamma_1, \ldots, \Pi_\ell$ of γ_ℓ with the obvious set inclusions

$$\Pi_\ell \subset \bar{\Pi}_\ell \subset \Pi_{\ell-1} \subset \bar{\Pi}_{\ell-1} \subset \cdots \subset \Pi_1 \subset \bar{\Pi}_1 \subset \Pi_0.$$

Also we can demand that each of these closed tubes $\bar{\Pi}_g$ is compact, and has a meridianal diameter less than $1/2^g$, for $0 \leq g \leq \ell$, in terms of some convenient metric on M. Moreover we assume that γ_1, and hence Π_1, encircles Π_0 exactly b_0-times; γ_2, and hence Π_2, encircles Π_1 exactly b_1-times, and so forth. The integers $b_0, b_1, b_2, \ldots, b_{\ell-1}$ have been selected, in accord with the usable prime list of $a = (a_0, a_1, a_2, a_3, \ldots)$ following the procedure of the algorithm.

Now we shall find a nondegenerate elliptic orbit $\gamma_{\ell+1}$ encircling tube Π_ℓ exactly b_ℓ-times, for an appropriate integer $b_\ell \geq 2$. Again we use the algorithm to define $b_\ell = q_\ell s_0^{c_\ell}$ (in Case 1, or $b_\ell = q_\ell s_{\ell+1}$ in Case 2), where q_ℓ is the first remaining usable prime in $a = (a_0, a_1, a_2, \ldots)$ after the previous stages have been completed. As before, $q_\ell \geq 2$ and the tube Π_ℓ give rise to an integer which is to be exceeded by $s_0^{c_\ell}$ (in Case 1, or by $s_{\ell+1}$ in Case 2). Then Theorem 7 guarantees the existence of the required elliptic orbit $\gamma_{\ell+1}$ of $dH^\#$ b_ℓ-encircling the tube Π_ℓ, and possessing distinct characteristic frequencies $0 < \omega_2^{\ell+1}, \ldots, \omega_n^{\ell+1} < 1/2$.

The choice of the tube $\Pi_{\ell+1}$, with $\bar{\Pi}_{\ell+1} \subset \Pi_\ell$, around $\gamma_{\ell+1}$ is now easy, so that the meridianal diameter is less than $1/2^{\ell+1}$.

In this way the sequence of nondegenerate elliptic orbits $\gamma_0, \gamma_1, \gamma_2, \gamma_3, \ldots$ is found within the corresponding tubular neighborhoods $\Pi_0 \supset \Pi_1 \supset \Pi_2 \supset \Pi_3 \supset \cdots$. The completed induction argument shows that $\gamma_{\ell+1}$ encircles the tube Π_ℓ exactly b_ℓ-times before closing periodically. The integers $b_\ell \geq 2$ are selected with reference to $a = (a_0, a_1, a_2, a_3, \ldots)$ in accord with the arithmetic algorithm leading to Lemma 1 before Theorem 7.

iv) Σ_b *homeomorphic to* Σ_a.

The abstract topological solenoid Σ_a with parameter $a = (a_0, a_1, a_2, a_3, \ldots)$ is prescribed in the theorem. In the preceding step we use the arithmetic algorithm for determining a sequence of integers $b = (b_0, b_1, b_2, b_3, \ldots)$ that specify an abstract topological solenoid Σ_b. But by Lemma 1 above, Σ_b is homeomorphic to Σ_a.

Noting that Σ_b and Σ_a are merely different parametric descriptions of the same topological solenoid, we caution that the proof has not yet established that this solenoid lies embedded in M as a minimal set for the flow $dH^\#$.

v.) *Embedding of* Σ_a *in* M.

Consider the sequence of nondegenerate elliptic orbits γ_0, γ_1, γ_2, γ_3, ... for $dH^\#$, each lying within the corresponding compact tube $\bar{\Pi}_0 \supset \bar{\Pi}_1 \supset \bar{\Pi}_2 \supset \cdots$, as constructed above in Step iii). Consider the nonempty compact set $\bigcap^{\infty}_{\ell=0} \bar{\Pi}_\ell$ in M. We shall show that Σ_a is homeomorphic with $\bigcap^{\infty}_{\ell=0} \bar{\Pi}_\ell$, and accordingly we refer to this compact set as a topological embedding of the solenoid Σ_a in M.

To verify this assertion we shall define a topological map F from the abstract solenoid Σ_b onto the intersection $\bigcap^{\infty}_{\ell=0} \bar{\Pi}_\ell$ in M. Since these techniques are routine in general topology, we shall omit some of the details of our construction of F.

Recall that Σ_b is defined as a subset of the countable self-product of the unit circle S^1 of the complex plane. In detail Σ_b consists of all points

$$\mathbf{z} = (z_0, z_1, z_2, z_3, \ldots) \in \Sigma_b \subset S^1 \times S^1 \times S^1 \times S^1 \times \cdots,$$

where each complex coordinate $z_\ell = e^{2\pi i \psi_\ell}$ for some frequency $0 \leq \psi_\ell < 1 \pmod 1$, and

$$z_0 = z_1{}^{b_0}, \qquad z_1 = z_2{}^{b_1}, \ \ldots, z_l = z_{\ell+1}^{b_\ell}, \ \ldots.$$

We must specify the corresponding point $F(\mathbf{z})$ in $\bigcap^{\infty}_{\ell=0} \bar{\Pi}_\ell$.

In order to simplify the definition of the map F we first introduce a longitude coordinate $0 \leq \psi < 1$ around the tube Π_0, with the section Σ corresponding to $\psi = 0$. Next modify the "speed" along the trajectories of $dH^\#$ in Π_0 so that along the modified flow $d\psi/dt \equiv 1$. This result can be obtained by multiplying the vector field $dH^\#$ by an appropriate C^k-function that is positive in Π_0, and such modifications do not affect the geometry of the periodic orbits of $dH^\#$. Henceforth we assume $d\psi/dt \equiv 1$ in Π_0.

The transversal section Σ, crossing γ_0 in Π_0, contains the initial points for each of the elliptic orbits γ_0, γ_1, γ_2, γ_3, ... For instance, γ_1 is located on a fixed-point of the b_0-th iterate of the Poincaré map around γ_0, and we select one of these fixed-points to mark a specific initial point for γ_1 in Σ. Then the transversal to γ_1 across the corresponding tube Π_1 can be taken as a $(2n - 1)$-ball in Σ. Similarly this transversal section across Π_1 contains the initial point of γ_2, and its corresponding transversal $(2n - 1)$-ball across Π_2. In this way each of the elliptic orbits γ_0, γ_1, γ_2, γ_3, ... has a designated initial point on Σ, and the corresponding transversals form a nested sequence of balls in Σ. Of course, γ_1 meets the section Σ in exactly b_0-points while completing its full orbit, and γ_2 meets Σ in $b_0 b_1$-points, etc. Moreover by using the modified trajectory speed for $dH^\#$ wherein $d\psi/dt \equiv 1$, we can assert that γ_0 has least period 1, γ_1 has least period b_0, γ_2 has least period $b_0 b_1$, etc.

With these geometric preliminaries out of the way we now return to the definition of the image point $F(\mathbf{z})$ in $\bigcap^\infty_{\ell=0} \bar{\Pi}_\ell$. Take the first coordinate $z_0 = e^{2\pi i \psi_0}$ of the point $\mathbf{z}$ in the solenoid Σ_b. For simplicity of exposition we take $\psi_0 = 0$, otherwise we would translate the section Σ along the flow $dH^\#$ by the longitude angle ψ_0. Hence the initial point of γ_0 in $\bar{\Pi}_0 \cap \Sigma$ lies at the frequency $\psi_0 = 0$ of the coordinate z_0. Next take $z_1 = e^{2\pi i \psi_1}$ and follow the orbit γ_1 in its tube Π_1 for a time duration $b_0\psi_1$ out from the initial point of γ_1 in $\bar{\Pi}_1 \cap \Sigma$. Since $\psi_0 = b_0\psi_1 \pmod 1$, we note that our trajectory speed modifications have forced $b_0\psi_1 = 0$, or 1, or 2, $\ldots$, or $(b_0 - 1)$. That is, we specify a point on γ_1 in $\bar{\Pi}_1 \cap \Sigma$, and thereby we specify one of the b_0 compact $(2n - 1)$-balls in Σ that is the corresponding component of $\bar{\Pi}_1 \cap \Sigma$. Similarly we next specify the point on the orbit γ_2 corresponding to duration $b_0 b_1 \psi_2 = 0$, or 1, or 2, $\ldots$, or $(b_0 b_1 - 1)$, and thus specify a compact component of $\bar{\Pi}_2 \cap \Sigma$.

Continue this selection of the nested sequence of compact balls in Σ corresponding to components of $\bar{\Pi}_\ell \cap \Sigma$. Since the diameters of these balls tend to zero, there is a unique point within their intersection and we designate this point as $F(\mathbf{z})$. In this way the map

$$F : \Sigma_b \to \bigcap_{\ell=0}^{\infty} \bar{\Pi}_\ell$$

is defined.

Clearly F is continuous from Σ_b into the manifold M. This follows because small changes in $\mathbf{z}$, say small in the first components $(z_0, z_1, z_2, z_3, \ldots, z_\ell)$, can change the image point $F(\mathbf{z})$ only within some short arc of the tube $\bar{\Pi}_\ell$. That is, the longitude of $F(\mathbf{z})$ varies only slightly within the tube $\bar{\Pi}_\ell$ which has a very small meridianal diameter if ℓ is large enough.

Finally we shall show that F is surjective and bijective onto the compact set $\bigcap^\infty_{l=0} \bar{\Pi}_l$. Take a point Q_∞ in $\bigcap^\infty_{\ell=0} \bar{\Pi}_\ell$, and for simplicity assume that the longitude of Q_∞ is $\psi = 0$ so $Q_\infty \in \Sigma$ (otherwise we translate the section Σ by the longitude of Q_∞). Take the initial coordinate $z_0 = e^{2\pi i \psi_0}$, for some point $\mathbf{z} = (z_0, z_1, z_2, z_3, \ldots)$, to be determined by requiring $\psi_0 = 0$. Now Q_∞ lies in $\bar{\Pi} \cap \Sigma$ and hence Q_∞ lies within some component of this compact set, as determined by the point on γ_1 at duration $b_0\psi_0 = 0$, or $1, \ldots$, or $(b_0 - 1)$. Take $\psi_1 = 0$, $1/b_0$, $2/b_0$, $\ldots$, or $(b_0 - 1)/b_0$ accordingly and then fix the next coordinate $z_1 = e^{2\pi i \psi_1}$. Continue to locate Q_∞ in $\bar{\Pi}_2 \cap \Sigma$ by means of the numbered components $0, 1, 2, \ldots, (b_0 b_1 - 1)$ along the orbit γ_2, and then define $z_2 = e^{2\pi i \psi_2}$ where $b_0 b_1 \psi_2 = 0, 1, 2, \ldots$, or $(b_0 b_1 - 1)$. In this way we determine a point $\mathbf{z}$ in Σ_b with $F(\mathbf{z}) = Q_\infty$. Hence F is surjective onto $\bigcap^\infty_{\ell=0} \bar{\Pi}_\ell$.

In a similar fashion suppose $\mathbf{z}$ and $\hat{\mathbf{z}}$ in the solenoid Σ_b differ in some coordinate place, say $z_\ell \neq \hat{z}_\ell$ in the l-th coordinate. Say $z_\ell = e^{2\pi i \psi_\ell}$ with $\psi_\ell = 0$ and the corresponding frequency for $\hat{z}_\ell$ is $\hat{\psi}_\ell \neq 0$. If $b_0 \cdot b_1 \cdots b_{\ell-1} \hat{\psi}_\ell = 0 \pmod 1$, then $F(\mathbf{z})$ and $F(\hat{\mathbf{z}})$ have different longitude coordinates in Π_0. But even if $b_0 b_1 \cdots b_{\ell-1} \hat{\psi}_\ell = 0 \pmod 1$, then $F(\mathbf{z})$ and $F(\hat{\mathbf{z}})$ lie in a different components of the intersection $\bar{\Pi}_\ell \cap \Sigma$. Hence F is injective.

Therefore

$$F: \Sigma_b \to \bigcap_{\ell=0}^{\infty} \bar{\Pi}_\ell$$

is a continuous bijective map of the compact metric space $\Sigma_b = \Sigma_a$ onto the compact subset $\bigcap^\infty_{\ell=0} \bar{\Pi}_\ell$ in M. Thus F is a homeomorphism of the solenoid Σ_a with $\bigcap^\infty_{\ell=0} \bar{\Pi}_\ell$ in M. With this construction in mind, we shall refer to the set $\bigcap^\infty_{\ell=0} \bar{\Pi}_\ell$ as the solenoid Σ_a embedded in the manifold M.

vi) *Σ_a is minimal for $dH^\#$ in M.*

We first characterize $\Sigma_a = \bigcap^\infty_{\ell=0} \bar{\Pi}_\ell$ as the limit set of the sequence of elliptic orbits $\gamma_0, \gamma_1, \gamma_2, \gamma_3, \ldots$, that is,

$$\lim_{\ell \to \infty} \gamma_\ell = \bigcap_{\ell=0}^{\infty} \bar{\Pi}_\ell.$$

Here we understand that a point $P_\infty \in M$ lies in this limit set just in case there exists a sequence of points $P_\ell \to P_\infty$, with $P_\ell \in \gamma_\ell$.

In our construction of the solenoid $\bigcap^{\infty}_{\ell=0} \bar{\Pi}_{\ell}$ in Steps iii), iv), and v), we showed that a point $Q_{\infty} \in \bigcap^{\infty}_{\ell=0} \bar{\Pi}_{\ell}$ must lie in some nested sequence of compact $(2n - 1)$-balls in a section Σ (provided the longitude of Q_{∞} is zero, otherwise we translate Σ around Π_0 to the longitude of Q_{∞}). Furthermore each of these compact balls contains a point $P_{\ell} \in \gamma_{\ell}$. Since $P_{\ell} \to Q_{\infty}$ we conclude that

$$\lim_{\ell \to \infty} \gamma_{\ell} \supset \bigcap_{\ell=0}^{\infty} \bar{\Pi}_{\ell}.$$

Conversely, take a sequence of points $P_{\ell} \in \gamma_{\ell}$ with $P_{\ell} \to P_{\infty}$ in M. Then, for sufficiently large ℓ, the longitude coordinates of P_{ℓ} are nearly that of P_{∞}, say longitude zero for convenience. We next locate a point $P_{\ell}' \in \Sigma$ with longitude zero, and so that $P_{\ell}' \to P_{\ell}$ as $\ell \to \infty$. Then each such point P_{ℓ}' lies in a compact component of $\bar{\Pi}_{\ell} \cap \Sigma$, and so the sequence P_{ℓ}' converges to some limit point in $\bigcap^{\infty}_{\ell=0} \bar{\Pi}_{\ell}$. Thus

$$\lim_{\ell \to \infty} \gamma_{\ell} \subset \bigcap_{\ell=0}^{\infty} \bar{\Pi}_{\ell}.$$

Therefore we have characterized $\Sigma_a = \bigcap^{\infty}_{\ell=0} \bar{\Pi}_{\ell}$ as the limit set of the sequence of elliptic orbits $\gamma_0, \gamma_1, \gamma_2, \gamma_3, \ldots$.

It is now evident that Σ_a is an invariant set of the flow $dH^{\#}$, as follows easily by standard continuity arguments of topological dynamics. In order to prove that Σ_a is a minimal set for $dH^{\#}$, we must show that each point, say the point P_{∞}, has a trajectory that is dense in Σ_a.

Pick the point P_{∞} in Σ_a, and pick any open set $\mathcal{O}$ of M that meets Σ_a. Take a point $Q \in \mathcal{O} \cap \Sigma_a$ and note that Q lies within every tube $\bar{\Pi}_{\ell}$. Then there must be some large integer ℓ_1, such that an open arc of Π_{ℓ_1} is contained in $\mathcal{O}$. But the trajectory of $dH^{\#}$ initiating at P_{∞} encircles throughout the entire tube Π_{ℓ_1} in a finite time duration, and hence this trajectory must meet $\mathcal{O}$.

Hence each trajectory of $dH^{\#}$ initiating at a point of Σ_a must be dense in Σ_a. Thus Σ_a is a minimal set for the flow of $dH^{\#}$.

These Steps i) through vi) conclude the proof of our Principal Theorem. □

Some unresolved questions. Is the flow of $dH^{\#}$ on the solenoid Σ_a equicontinuous and almost periodic? Of course, the modified flow with $d\psi/dt \equiv 1$ has this behavior, but the exact demand of the equality casts doubt on the generic character of these properties.

In another direction of research we could discard the hypothesis that the symplectic manifold M should be compact. As remarked above Theorem 7, and its consequences for the Principal Theorem, are valid for noncompact M. The necessity of starting from an elliptic critical point, in the proof of the Principal Theorem, could be overcome by considering only Hamiltonian functions H that tends towards infinity near the boundary of M.

But more difficult questions arise if we consider a symplectic $2n$-manifold M that is presented as the cotangent bundle T^*N of a Riemannian n-manifold N. For each real function V on N we can form a Hamiltonian function $H = T + V$ on M, where T is the Riemann metric tensor on N interpreted on $M = T^*N$. This is the class of Hamiltonians that arise in classical mechanics where N is the configuration manifold of the dynamical problem and M is the momentum phase space. If we fix T and allow perturbations of the potential function V only, then new types of problems arise in the theory of generic Hamiltonians, see [9 p. 49]. In particular the theorems of Robinson and of Birkhoff are uncertain under these severe limitations, and hence the main body of our theory of generic Hamiltonians remains for future investigation, within this framework.

Another similar question arises from considering geodesic flows on a compact manifold. Klingenberg [6] has shown that a generic geodesic flow on a compact manifold has infinitely many closed geodesics but these geodesics might be all hyperbolic (for example if the manifold has constant negative curvature). Thus the methods of this paper would not really yield solenoidal minimal sets for generic geodesic flows.

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