HOMOCLINIC POINTS OF AREA PRESERVING DIFFEOMORPHISMS.

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Since their introduction by Poincaré, homoclinic points of diffeomorphisms have been the subject of considerable research. Homoclinic points naturally arise in the study of section maps defined by ordinary differential equations. Indeed Poincaré encountered them in his studies of various section maps defined by the equations of the restricted three body problem. The works of Poincaré [8], Birkhoff [2], Smale [9] and Alekseev [1] have shown that the existence of homoclinic points implies considerable complexity in the orbit structure of the diffeomorphism. However there have been very few concrete examples of diffeomorphisms which can be shown to have homoclinic points.

Cherry [3] gave an example of an analytic diffeomorphism obtained as the section map of a modified pendulum equation which has a nondegenerate homoclinic point. In Cherry's introduction he states that Poincaré's analysis of an earlier example was not complete and therefore Poincaré did not actually give an example of a homoclinic point. Thus Cherry's is the first published example of an analytic diffeomorphism with a non-degenerate homoclinic point. Smale [9] gave a simple geometric construction for C^∞ examples. Smale first constructs the diffeomorphism and then the differential equation which gives this diffeomorphism as a section map. McGehee [6] gave an example of the related concept of an orbit homoclinic to a periodic orbit in the restricted three-body problem. The work of Sitnikov [10] gives rise to another example of a homoclinic point in the three body problem. One can view Sitnikov's work as giving an example of an orbit homoclinic to a periodic orbit at infinity in the three body problem (see [7] for details).

The most important examples are those homoclinic points that can be shown to be nondegenerate (i.e., the stable and unstable manifolds intersect transversally). The construction of examples can be divided into two steps in

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general. First one shows that a homoclinic point exists and then one shows that the homoclinic point is nondegenerate. In many cases the first step is a geometric problem whereas the second is an analytic problem. In this paper we shall discuss the first step of establishing the existence of homoclinic points by geometric methods.

The main theorem of this paper (theorem 1) states that under some minor hypotheses homoclinic points are stable under small perturbations when the diffeomorphism is area preserving. This theorem is used in conjunction with the method of averaging to show that two variations of Duffing's equation have homoclinic orbits.

2. Notation and Main Theorem. Let (M,Ω) be a C^r, $r \geqslant 1$, two-dimensional symplectic manifold, i.e., M is a two-dimensional differentiable manifold and Ω is closed, nondegenerate two form on M. Since M is two dimensional Ω is simply an area form on M. Assume that M has

Property A: Every simple closed curve in M separates M into two regions one of which has finite area.

Let $\mathcal{F}$ denote the class of C^r diffeomorphisms $f: M \to M$, such that $df^*(\Omega) = \Omega$. An $f \in \mathcal{F}$ is called a symplectic or area preserving diffeomorphism. Give $\mathcal{F}$ the C^1 compact open topology.

If $f \in \mathcal{F}$ then $p \in M$ is said to be a homoclinic point of f if there exists a fixed point q of f, such that $p \neq q$ and $\lim_{n\to\infty} f^n(p) = \lim_{n\to\infty} f^{-n}(p) = q$. In this case we say that p is homoclinic to q under the action of f. A fixed point q of f is said to be a hyperbolic fixed point if $Df(q): T_qM \to T_qM$ has no eigenvalue of modulus 1. Since M is two dimensional and f is area preserving the eigenvalues of $Df(q)$ at a hyperbolic fixed point q must be of the form λ, λ^{-1}, where λ is a real number $\neq 0, 1$.

Let $\mathcal{H} \subset \mathcal{F}$ be the set of all $f \in \mathcal{F}$ such that some $p \in M$ is homoclinic to a hyperbolic fixed point q of f.

Our main theorem is:

THEOREM 1. *If M has property A then $\mathcal{H}$ is open in $\mathcal{F}$.*

Thus under the above assumptions homoclinic points (degenerate or nondegenerate) are stable under small perturbations. In fact we shall prove the stronger statement that if p is homoclinic to the hyperbolic fixed point q under the action of f then there exists a neighborhood N of q and a neighborhood $\mathcal{U}$ of f such that for each $g \in \mathcal{U}$ the map g has a unique hyperbolic fixed point $\bar{q} = \bar{q}(g)$ in N and some $\bar{p} \in M$ is homoclinic to $\bar{q}$ under the action of g. In the next section several modifications of the above theorem will be given along with two concrete applications. The idea behind the proof of theorem 1 is found in Poincaré's discussion of homoclinic points [8].

In order to prove theorem 1 precise knowledge about the local structure of f near a hyperbolic fixed point is needed. The local structure of a map near a hyperbolic fixed point has been studied extensively and a summary of the known results is given below.

Let q be a hyperbolic fixed point of $f \in \mathcal{F}$ and N a neighborhood of q. Then define

$$W^+(q,f) = \{ p \in M : f^n(p) \to q \text{ as } n \to \infty \},$$

$$W^-(q,f) = \{ p \in M : f^{-n}(p) \to q \text{ as } n \to \infty \},$$

$$W_N^+(q,f) = \{ p \in M : f^n(p) \in N \text{ for } n \geqslant 0 \text{ and } f^n(p) \to q \text{ as } n \to \infty \},$$

$$W_N^-(q,f) = \{ p \in M : f^{-n}(p) \in N \text{ for } n \geqslant 0 \text{ and } f^{-n}(p) \to q \text{ as } n \to \infty \},$$

as the stable, unstable, local stable and local unstable manifold (respectively) of q under the action of f. Clearly

$$W^+(q,f) = \bigcup_{n=0}^{\infty} f^{-n}(W_N^+(q,f))$$

and

$$W^-(q,f) = \bigcup_{n=0}^{\infty} f^{n}(W_N^-(q,f)).$$

THEOREM 2. *Let $q \in M$ be a hyperbolic fixed point of $f \in \mathcal{F}$ and let the eigenvalues of $Df(q)$ be λ, λ^{-1} where $0 < |\lambda| < 1$. Then there exists a C^1 coordinate chart $\varphi : N \to R^2$, N open in M, $\varphi(N) = (-2,2)^2$, $\varphi(q) = (0,0)$, such that*

$$\varphi \circ f \circ \varphi^{-1} : (-2,2) \times (-2|\lambda|, 2|\lambda|) \to (-2,2)^2 : (x,y) \to (\lambda x, \lambda^{-1} y).$$

Moreover there exists a neighborhood $\mathcal{U}$ of f in $\mathcal{F}$ such that for all $g \in \mathcal{U}$ the map g has a unique hyperbolic fixed point $\bar{q} = \bar{q}(g) \in N$ and there exist C^1 functions $u^+ = u^+(g)$ and $u^- = u^-(g)$ such that

$$u^- : (-2,2) \to (-2,2)$$

$$u^+ : (-2,2) \to (-2,2)$$

and

$$\varphi(W_N^+(\bar{q},g)) = \{(x,u^+(x)) : |x|<2\}$$

and

$$\varphi(W_N^-(\bar{q},g)) = \{(u^-(y),y) : |y|<2\}.$$

Also for any $\epsilon>0$ *the neighborhood* $\mathcal{U}$ *may be chosen so that*

$$|u^+(x)| + \left|\frac{du^+}{dx}(x)\right| < \epsilon \text{ for } |x|<2$$

and

$$|u^-(y)| + \left|\frac{du^-}{dy}(y)\right| < \epsilon \text{ for } |y|<2.$$

The first paragraph of the above is a theorem of Hartman [4]. The second and third paragraphs are a rewording of the classical stable manifold theorem. A complete proof including the estimates of the third paragraph can be found in Hartman's book [5].

Let $p_1, p_2 \in W^+(q,f)$ then for some $k \geqslant 0$ one has $f^k(p_i) \in W_N^+(q,f)$. Let l be the closed line segment on the x axis in $\varphi(N)$ which joins $p \circ f^k(p_1)$ and $\varphi \circ f^k(p_2)$. Then define $[p_1,p_2]^+$ to be the closed arc in $W^+(q,f)$ with end-points p_1 and p_2 given by $[p_1,p_2]^+ = f^{-k} \circ \varphi^{-1}(l)$. We make a similar definition for $[p_1,p_2]^-$ if $p_i \in W^-(q,f)$. Note that both definitions are independent of k.

Proof of Theorem 1. Let $p \in M$ be homoclinic to the hyperbolic fixed point q under the action of $f \in \mathcal{F}$. By considering f^2 if necessary we may assume that the eigenvalues of $Df(q)$ are λ, λ^{-1}, where $0<\lambda<1<\lambda^{-1}$. Let N, φ, $\mathcal{U}$ be as given in theorem 2 with ϵ to be chosen below. Then

$$\varphi(W_N^+(q,f)) = X = \{(x,0) : |x|<2\}$$

and

$$\varphi(W_N^-(q,f)) = Y = \{(0,y) : |y|<2\}.$$

Since p is homoclinic to q, $p \in W^+(q,f) \cap W^-(q,f) - \{q\}$. Thus there exist non negative integers k_1 and k_2, such that $f^{k_1}(p) \in W_N^+(q,f)$ and $f^{-k_2}(p) \in W_N^-(q,f)$. Then $\varphi(f^{k_1}(p)) = z \in X$ and $\varphi(f^{-k_2}(p)) = w \in Y$. Moreover we may choose k_1 and k_2 so that if $z=(x_0,0)$ and $w=(0,y_0)$ then $0<|x_0|<1$ and $0<|y_0|<1$. For simplicity we shall take x_0 and y_0 positive. Define $k=k_1+k_2$ and $F = \varphi \circ f^{-k} \circ \varphi^{-1}$ so $F(z)=w$ (see Figure 1).

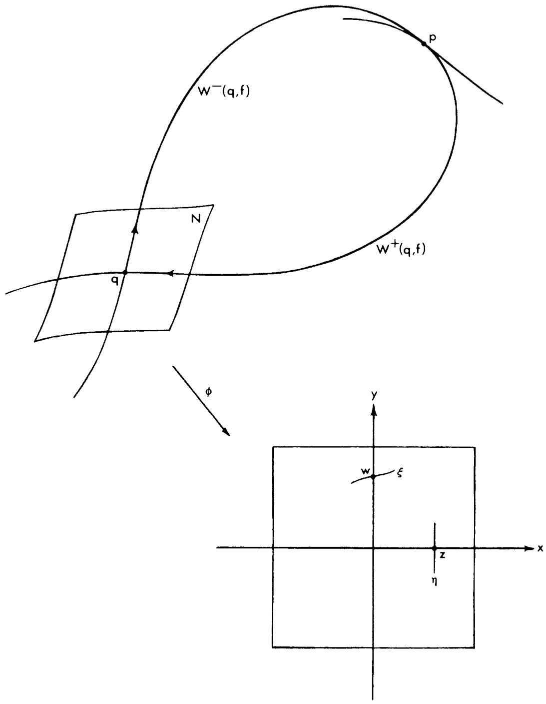

Figure 1

There are two cases to consider.

Case I. $DF(z)(T_zX) \neq T_wY$ (the nondegenerate homoclinic point). Even though this case is said to follow from the general transversality theorem a proof will be presented for completeness. The above condition states that the image of X under F is not tangent to Y at the point of intersection w. Thus there exists a $\delta > 0$ such that $0 < x_0 - \delta < x_0 + \delta < 1$ and the image of $\{(x,0): |x - x_0| \leqslant \delta\}$ is an arc in $(-1,1)^2$ joining $v_1 = (x_1, y_1)$ to $v_2 = (x_2, y_2)$ with $x_1 > 0$ and $x_2 < 0$. Let $\alpha > 0$ and $\mathcal{U}$ be so small that if $g \in \mathcal{U}$ and $G = \varphi \circ g^{-k} \circ \varphi^{-1}$ then G maps the segments $\{(x_0 - \delta, y): |y| \leqslant \alpha\}$ and $\{(x_0 + \delta, y): |y| \leqslant \alpha\}$ into two arcs one of which lies in $\{(x,y) \in (-1,1)^2 : x < \frac{1}{2}x_2\}$ and the other lies in $\{(x,y) \in (-1,1)^2 : x > \frac{1}{2}x_1\}$.

Now let $\mathcal{U}$ be chosen so small that for all $g \in \mathcal{U}$ the ϵ of theorem 2 is less than $\frac{1}{2}\min(\alpha, x_1, -x_2)$ and the x coordinate of $\varphi(\bar{q})$ is less than $x_0 - \delta$. Thus the segment γ of $\varphi(W_N^+(\bar{q}, g))$ above $[x_0 - \delta, x_0 + \delta]$ (i.e. $\gamma = \{(x, u^+(x)): |x - x_0| < \delta\}$ does not contain $\varphi(\bar{q})$ and is mapped by G onto an arc in $(-1,1)^2$ joining two points $v_3 = (x_3, y_3)$ and $v_4 = (x_4, y_4)$ with $x_3 > \frac{1}{2}x_1 > 0$ and $x_4 < \frac{1}{2}x_2 < 0$. But

$\varphi(W_N^-(\bar{q},g))$ is an arc in $(\frac{1}{2}x_2,\frac{1}{2}x_1)\times(-2,2)$ which meets the segments $[\frac{1}{2}x_2,\frac{1}{2}x_1]\times\{1\}$ and $[\frac{1}{2}x_2,\frac{1}{2}x_1]\times\{-1\}$. Thus $G(\gamma)$ meets two opposite sides of $[\frac{1}{2}x_2,\frac{1}{2}x_1]\times[-1,1]$ and $\varphi(W_N^-(\bar{q},g))$ meets the other two opposite sides and so there is a point $\varphi(h)$ in common. Since $\varphi(\bar{q})\not\in\gamma$ we know that $h\neq\bar{q}$. Since $\varphi(h)\in\varphi(W_N^-(\bar{q},g))$ we have $h\in W^-(\bar{q},g)$ and since $\varphi(h)\in G(\gamma)$ $=\varphi\circ g^{-k}\circ\varphi^{-1}(\gamma)\subset\varphi g^{-k}(W_N^+(\bar{q},g))$ we have $h\in g^{-k}(W_N^+(\bar{q},g))\subset W^+(\bar{q},g))$. Thus h is a homoclinic point of g. A more careful analysis will show that h is a nondegenerate homoclinic point also.

Case II. $DF(z)(T_zX)=T_wY$ (the degenerate homoclinic point).

Let $\eta=\{(x_0,y):|y|\leqslant\delta\}$ and $F(\eta)=\xi$. So η and ξ are closed arcs in $(-1,1)^2$ and $z\in\operatorname{int}\eta$, $w\in\operatorname{int}\xi$ for δ small. It is clear that $\varphi\circ f\circ\varphi^{-1}(\eta)\cap\eta=\varnothing$. Since η intersects X transversally at z and $DF(z)(T_zX)=T_wY$ it follows that ξ intersects Y transversally at w. Let $S(\alpha)=\{(x,y)\in R^2:|x|\leqslant\alpha|y|\}$. Then by choosing δ small we may assume that $T_r\xi\cap S(\alpha)=\{0\}$ for some $\alpha>0$ and all $r\in\xi$. That is ξ is the graph of a function of x for small x and the derivative of this function is bounded above and below by $\pm\alpha^{-1}$.

Now choose $\mathcal{U}$ so small that for all $g\in\mathcal{U}$:

1. g has a hyperbolic fixed point $\bar{q}\in N$,
2. $\xi'=\xi'(g)=G(\eta)\subset(-1,1)^2$, where, as before, $G=\varphi\circ g^{-k}\circ\varphi^{-1}$,
3. $T_r\xi'\cap S(\alpha/2)=\{0\}$ for all $r\in\xi'$,
4. $\varphi(W_N^+(\bar{q},g))$ intersects $\operatorname{int}\eta$ in one point $\bar{s}$ (i.e. $\epsilon<\delta$),
5. $\varphi(W_N^-(\bar{q},g))$ intersects $\operatorname{int}\xi'$ in one point $\bar{t}$ and $|du^-/dy(y)|<\alpha/2$ for $|y|<2$ where u^- is the function whose graph is $\varphi(W_N^-(\bar{q},g))$ (i.e., $\epsilon<\alpha/2$), and
6. $g\circ\varphi^{-1}(\eta)\cap\eta=\varnothing$.

Now let $\hat{l}=g^{-k_1}\circ\varphi^{-1}(\eta)=g^{k_2}\circ\varphi^{-1}(\xi')$ and $s=g^{-k_1}\circ\varphi^{-1}(\bar{s})$, $t=g^{k_2}\circ\varphi^{-1}(\bar{t})$. Thus $s,t,\in\hat{l}$. Now assume that g has no homoclinic points so $s\neq t$. Let l be the closed segment in $\hat{l}$ whose endpoints are t and s. (See Figure 2). By 6 $g(l)\cap l=\varnothing$.

Then $C_1=[\bar{q},t]^-\cup l\cup[\bar{q},s]^+$ is a simple closed curve and hence bounds two regions one of which, say R, has finite area. Also $g(C_1)=C_2$ $=[\bar{q},g(t)]^-\cup g(l)\cup[\bar{q},g(s)]^+$ is a simple closed curve bounding $g(R)$.

Either $l-\{s,t\}$ lies in $g(R)$ or in $M-g(R)$.

Assume that $l-\{s,t\}$ lies in $g(R)$; in the other case proceed with the following argument replacing g by g^{-1}. The simple closed curve C_3 $=l\cup[s,g(s)]^+\cup g(l)\cup[g(t),t]^-$ is made up of four arcs the first two of which have their interiors in $g(R)$ and the last two are part of $g(C_1)=C_2$. Thus C_3 bounds a region Q which lies in $g(R)$ and not in R. Thus $g(R)\supset R\cup Q$ but this contradicts the fact that g is area preserving since Q has nonzero area.

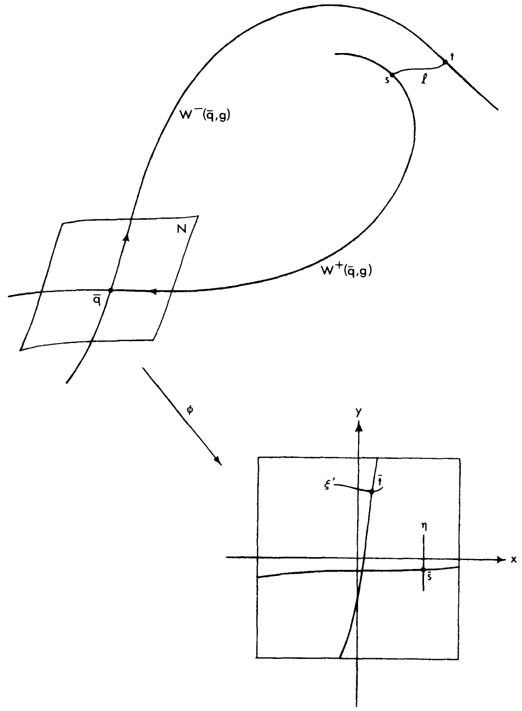

Figure 2

Remark. A review of the proof of the above theorem shows that not every simple closed curve in M need separate M into two regions one of which has finite area. One only needs that $[q,p]^{+} \cup [q,p]^{-}$ is a simple closed curve which separates M into two regions one of which has finite area.

3. Homoclinic Points in Averaged Systems. Degenerate homoclinic points often occur in averaged Hamiltonian systems and so in this section we shall show that these homoclinic points persist under small perturbations. This extension of theorem 1 is then applied to Duffing's equation.

Consider the Hamiltonian

$$H(t,x,\epsilon) = \epsilon H_1(t,x) \tag{1}$$

where H_1 is a C^r, $r \geq 2$, function for all $t \in R$, $x \in R^2$ and H_1 is T periodic in t. Let the corresponding Hamiltonian equations

$$\dot{x}^1 = \frac{\partial H}{\partial x^2}, \qquad \dot{x}^2 = \frac{\partial H}{\partial x^1} \tag{2}$$

have solutions $\varphi(t,y,\epsilon)$ where $\varphi(0,y,\epsilon) = y$.

Let $\Psi(y,\epsilon)=\varphi(T,y,\epsilon)$ be the period map defined by the solutions of (2). It is clear that for any $M>0$ there exists an $\epsilon_1>0$ such that $y\to\Psi(y,\epsilon)$ is a well-defined diffeomorphism of $\{y\in R^2:|y|<M\}$ into R^2 (for $|\epsilon|<\epsilon_1$ and that $\Psi(y,\epsilon)=y+0(\epsilon)$ uniformly in y for $|y|<M$. Since Ψ is the period map defined by a Hamiltonian system Ψ is area preserving. However since Ψ is a perturbation of the identity one cannot hope to find a homoclinic point by theorem 1 directly. However:

COROLLARY 1. *Let the function* $H_0(x)=(1/T)\int_0^T H_1(s,x)ds$ *have a nondegenerate saddle point at* x_0 *and assume that the level set* $\{x\in R^2:H_0(x)=H_0(x_0)\}$ *contains a simple closed curve* C *such that* $x_0\in C$. *Then there exists an* $\epsilon_1>0$ *and points* $p(\epsilon)$, $q(\epsilon)\in R^2$ *for* $0<|\epsilon|<\epsilon_1$ *such that* $p(\epsilon)$ *is homoclinic to the hyperbolic fixed point* $q(\epsilon)$ *under the action of* Ψ *and* $q(\epsilon)=x_0+0(\epsilon)$.

Proof. Define a symplectic change of variables $x\to u$ by

$$x^1=\frac{\partial S}{\partial x^2}, \qquad u^2=\frac{\partial S}{\partial u^1} \tag{3}$$

where

$$S(t,u^1,x^2,\epsilon)=u^1x^2+\epsilon\int_0^t\{H_1(s,u^1,x^2)-H_0(u,x^2)\}ds. \tag{4}$$

Since H_0 is the average of H_1 over a period S, defines a change of variables that is T periodic in t. Moreover for any $M>0$ there is an $\epsilon_0>0$ such that (3) is a well-defined diffeomorphism for $|x|<M$ or $|u|<M$ when $|\epsilon|<\epsilon_0$ since the map (3) is the identity map when $\epsilon=0$. In the new variables the Hamiltonian becomes

$$H(t,u,\epsilon)=\epsilon H_0(u)+O(\epsilon^2), \tag{5}$$

where $O(\epsilon^2)$ is uniform in u for $|u|<M$.

Let M be so chosen that $C=\{x:|x|<M\}$. In the differential equations defined by the Hamiltonian (5) make the change of time $\tau=\epsilon t$ so the equations become

$$\frac{du^1}{d\tau}=\frac{\partial H_0}{\partial u^2}(u)+0(\epsilon), \qquad \frac{du^2}{d\tau}=-\frac{\partial H_0}{\partial u^1}(u)+0(\epsilon), \tag{6}$$

where these equations are now ϵT periodic in τ. Since H_0 has a nondegenerate saddle point at $u=x_0$ the implicit function theorem gives that the equations (6) have a hyperbolic periodic solution of period ϵT for ϵ small and when $\tau=0$ this periodic solution has initial condition $x_0+0(\epsilon)$. Let $n=[1/\epsilon]$, i.e., the greatest integer in $1/\epsilon$, for $\epsilon \neq 0$. If $\xi(t,\eta,\epsilon)$ is the solution of (6) with $\xi(0,\eta,\epsilon)=\eta$ then the periodic map $f_\epsilon:\eta \to \xi(n\epsilon T,\eta,\epsilon)$, $\epsilon \neq 0$, has a hyperbolic fixed point at $x_0+0(\epsilon)$. When $\epsilon=0$ define $f_0:\eta\to\xi(T,\xi,\epsilon)$.

Since equations (6) admit H_0 as an integral when $\epsilon=0$ and the level set $\{u:H_0(u)=H_0(x_0)\}$ contains a simple closed curve C, $x_0 \in C$, the mapping f_0 has a homoclinic point q which is homoclinic to x_0. Theorem 1 now applies since f_ϵ is a small perturbation of f_0 in the C^1 compact open topology and proves the corollary. We note that the statement of theorem 1 required that the mappings be globally defined but a check of the proof only requires that the mappings be defined in a neighborhood of the stable and unstable manifolds of the unperturbed map. This is true in the present case since $C \subset \{u:|u|<M\}$.

Example 1. Consider Duffing's equation

$$\ddot{v}+v+\epsilon\{2\alpha v+4\beta v^3+\gamma\cos t\}=0. \tag{7}$$

Note that the natural frequency $\sqrt{1-2\alpha\epsilon}$ is close to the forcing frequency 1 when ϵ is small. The equation (7) is equivalent to the Hamiltonian system

$$\dot{v}^1=\frac{\partial K}{\partial v^2}=v^2,$$

$$\dot{v}^2=\frac{-\partial K}{\partial v^1}=v^1-\epsilon\left\{2\alpha v^1+4\beta(v^1)^3+\gamma\cos t\right\}, \tag{8}$$

where

$$K=\tfrac{1}{2}\left\{(v^1)^2+(v^2)^2\right\}+\epsilon\left\{\alpha(v^1)^2+\beta(v^1)^4+\gamma v^1\cos t\right\}. \tag{9}$$

Now make the change of variables

$$\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}=\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{10}$$

to get the new Hamiltonian system

$$\dot{x}^1=\frac{\partial H}{\partial x^2}, \qquad \dot{x}^2=\frac{-\partial H}{\partial x^1} \tag{11}$$

where

$$H=\epsilon\left\{\alpha(x^1\cos t+x^2\sin t)^2+\beta(x^1\cos t+x^2\sin t)^4+\gamma(x^1\cos t+x^2\sin t)\cos t\right\}. \tag{12}$$

The Hamiltonian (12) is of the form (1) and so one computes that the average of H is ϵH_0 where

$$H_0(x^1,x^2) = (\alpha/2)\{(x^1)^2 + (x^2)^2\} + (3\beta/8)\{(x^1)^2 + (x^2)^2\}^2 + (\gamma/2)x^1. \quad (13)$$

When α and β have the same sign H_0 has only one critical point which is either a maximum or a minimum and so corollary 1 does not apply. Assume α and β have opposite signs, say $\beta > 0$ and $\alpha < 0$. Then if $|\alpha|^{3/2} > (9/8)|\gamma|\beta^{1/2} > 0$ the function H_0 has three critical points; two minimums and a non degenerate saddle point. A plot of the level lines of H_0 is shown in figure 3 and one sees that the conditions of corollary 1 are satisfied. Thus if $\beta > 0$, $\alpha < 0$, and $|\alpha|^{3/2} > (9/8)|\alpha|\beta^{1/2} > 0$, the period map defined by Duffing's equation (7) has a homoclinic point for ϵ small.

The next example is slightly different. A $q \in M$ is called a periodic point of $f \in \mathcal{F}$ of least period l if $f^l(q) = q$ and l is the least positive integer such that $f^l(q) = q$. In this case the orbit of q, $\mathcal{O}(q) = \{q, f(q), \ldots, f^{l-1}(q)\}$ consists of l distinic points. The periodic point q is called a hyperbolic periodic point if q is a hyperbolic fixed point of f. A point $p \in M$ is called a homoclinic point (homoclinic to $\mathcal{O}(q)$) if $p \not\in \mathcal{O}(q)$ and $\lim_{n\to\infty} f^n(l) = \lim_{n\to\infty} f^{-n}(l) = \mathcal{O}(q)$.

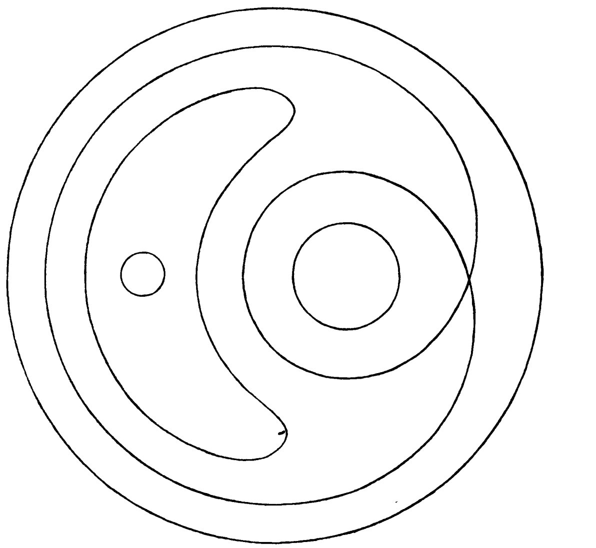

Figure 3

With some minor technical changes in the proofs theorem 1 and corollary 1 can be extended to cover this case also. With this in mind let us look at a different form of Duffing's equation.

Example 2. Consider

$$\ddot{v}+v+\epsilon\{2\alpha v+4\beta v^3\}+8\gamma\cos 3t=0. \tag{14}$$

When $\epsilon=0$ this equation has a unique $2\pi/3$ periodic solution $\gamma\cos 3t$. Let $v=u+\gamma\cos 3t$ then u satisfies the equation

$$\ddot{u}+u+\epsilon\left\{2\alpha(u+\gamma\cos 3t)+4\beta(u+\gamma\cos 3t)^3\right\}=0. \tag{15}$$

Equation (15) is equivalent to the Hamiltonian

$$\dot{u}^1=\frac{\partial K}{\partial u^2},\qquad \dot{u}^2=\frac{-\partial K}{\partial u^1}, \tag{16}$$

where

$$K=\tfrac{1}{2}\left\{(u^1)^2+(u^2)^2\right\}+\epsilon\left\{\alpha(u^1+\gamma\cos 3t)^2+\beta(u^1+\gamma\cos 3t)^4\right\}. \tag{17}$$

Again make the change of variables

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix}=\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{18}$$

to get the new Hamiltonian system

$$\dot{x}^1=\frac{\partial H}{\partial x^2},\qquad \dot{x}^2=\frac{-\partial H}{\partial x^1}, \tag{19}$$

where

$$H=\epsilon\left\{\alpha(x^1\cos t+x^2\sin t+\gamma\cos 3t)^2+\beta(x^1\cos t+x^2\sin t+\gamma\cos 3t)^4\right\}. \tag{20}$$

One computes the average of this function to be

$$H_0(x^1,x^2)=aI+bI^2+cI^{3/2}\cos 3\varphi, \tag{21}$$

where $I=\frac{1}{2}\{(x^1)^2+(x^2)^2\}$, $\varphi=\tan^{-1}x^2/x^1$, $a=\alpha+3\gamma^2\beta$, $b=3\beta/2$, and $c=\sqrt{3\gamma}$.

If a, b, and c are nonzero and $9c^2>32ba$ then (21) has 7 critical points. Three of these critical points are hyperbolic and they correspond to a periodic point of period 3 of the period map defined by equation (14). One can plot the

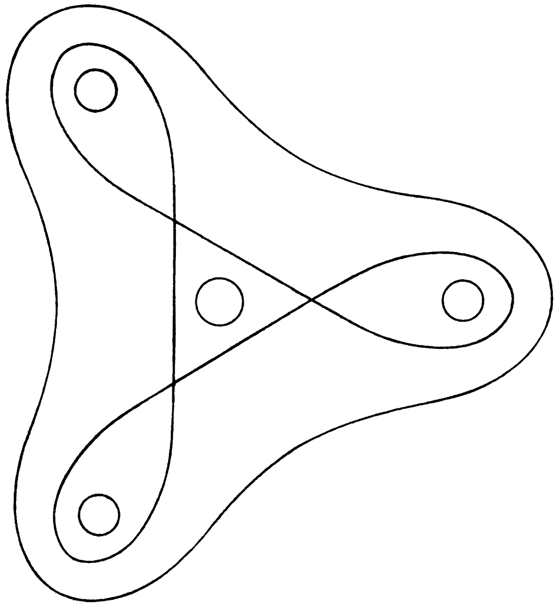

Figure 4

level lines of (21) (see figure 4) and see that the stable and unstable manifolds of these periodic points coincide. Thus there are homoclinic points for the section map defined by (17) which are homoclinic to a periodic point of least period 3 for ϵ small when a, b, c are nonzero and $9c^2 > 32ba$.

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