SIAM REVIEW
Vol. 28, No. 1, March 1986

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COUNTER-EXAMPLES IN DYNAMICAL SYSTEMS VIA NORMAL FORM THEORY*

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Abstract. The theory of normal forms is used to construct examples and counter-examples in the theory of ordinary differential equations. Selected examples from linearization theory, stability theory, bifurcation theory and theory of integrability are given.

Key words. Ordinary differential equations, normal forms, linearization, stability, bifurcation, invariant tori

AMS(MOS) subject classifications. Primary 34-02; secondary, 34C15, 34C29, 34A34

1. Introduction. A fully developed mathematical theory contains not only definitions, lemmas and theorems but also a good store of examples and counter-examples. In general the construction of a new example requires considerable ingenuity on the part of the researcher, but once constructed it can often be modified to create other examples which illustrate other points. Thus, examples often fall into groups.

In the theory of ordinary differential equations one such group of examples arises as a natural consequence of the theory of normal forms. These examples are easy to analyze, since usually equations in normal form can be explicitly integrated. By the nature of the theory of normal forms these examples illustrate the effect of higher order terms on the flow.

In §2 a brief survey of the main results about normal forms is given. In the sections that follow, examples from various areas of differential equations are given. In each section it is shown how the theory of normal forms is used to select the example. I hope that by the end the reader will be able to construct a variety of examples for himself. I have selected examples from linearization theory, stability theory, bifurcation theory and the theory of nonintegrability. None of the examples are new, but I believe that there is something new in each presentation.

I would like to thank Mr. Dan Mack for the fine drawings of the phase portraits.

2. Background on normal forms. The theory of normal forms arises from perturbation analysis and thus is usually presented for equations which contain a small parameter. In many cases, the small parameter is introduced by scaling and therefore is simply an aid in the analysis. Thus, one considers an equation of the form

$$\dot{z} = Z_*(z,\varepsilon) \tag{1}$$

where Z_* has a series expansion in ε of the form

$$Z_*(z,\varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j Z_j(z). \tag{2}$$

When $\varepsilon = 0$, (1) reduces to

$$\dot{z} = Z_0(z), \tag{3}$$

*Received by the editors May 1, 1984, and in revised form July 2, 1985.
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so (1) is to be considered as a perturbation of (3). In general (3) is simple (i.e., integrable or linear) and one wants to know what properties of (3) are preserved when $\varepsilon \neq 0$. The method of attack is to construct a near-identity change of variables $z = z(\zeta, \varepsilon) = \zeta + \cdots$ which reduces (1) to

$$\dot{\zeta} = Z^*(\zeta, \varepsilon) \tag{4}$$

where

$$Z^*(\zeta, \varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j Z^j(\zeta). \tag{5}$$

The series for the change of variables is constructed order by order so as to make (4) as simple as possible. Since $z = \zeta$ for $\varepsilon = 0$, (4) is a perturbation of (3) also, i.e., $Z^0 = Z_0$.

The basic philosophy is to try to eliminate as many terms as possible in (5) so the terms which remain in Z^* are those which cannot be transformed away. Therefore, the terms in Z^* must have an important effect on the flow generated by (4). So, if you wish to construct an example which shows that a perturbation of (3) destroys some property of the flow defined by (3) then a logical first choice is a perturbation which is already in normal form.

A great deal of the theory of normal forms is formal. That is, the questions of convergence remain unanswered. That will not affect the questions addressed here because only examples are sought. What is important is the nature of the terms in Z^*, or what is the form of the normal form. One of the standard theorems which answers this question is

THEOREM. *Let* $\{P_i\}$, $i = 0, 1, \cdots$ *be a sequence of vector spaces of smooth vector fields defined on a common domain such that*

(i) $Z_i \in P_i$ *for* $i \geq 0$,
(ii) $[P_i, P_j] \subset P_{i+j}$ *for* $i, j \geq 0$,
(iii) *the operator* $L_i = [\cdot, Z^0]$: $P_i \to P_i$: $A \to [A, Z^0]$ *is split surjective, i.e.,* $P_i = (\text{kernel } L_i) \oplus (\text{range } L_i)$ *for* $i \geq 0$.

Then there is a formal change of variables $z = \zeta + \cdots$ *which transforms* (1) *to* (4) *where* $[Z^i, Z^0] = 0$ *for* $i \geq 0$.

In the above $[\cdot, \cdot]$ is the Lie bracket operator defined by

$$[U, V] = \frac{\partial U}{\partial x} V - \frac{\partial V}{\partial x} U$$

where U and V are smooth vector fields depending on x. A simple calculation shows that $[U, V]$ is a vector field also. A classical theorem states that $[U, V] = 0$ if and only if the flows defined by U and V commute. That is, let $\phi_t(\xi) = \phi(t, \xi)$ (respectively $\psi_t(\xi) = \psi(t, \xi)$) be the solution of $\dot{x} = U(x)$ (resp. $\dot{x} = V(x)$) which satisfies $\phi(0, \xi) = \xi$ (resp. $\psi(0, \xi) = \xi$). Then $[U, V] = 0$ if and only if $\phi_s \circ \psi_t = \psi_t \circ \phi_s$ for all values of t and s for which this formula is defined.

Roughly speaking, the idea behind the proof of the above theorem is as follows. First the change of variables is constructed order by order. Let W be the kth term in the expansion of z, i.e., $z(\zeta, \varepsilon) = \zeta + \cdots + \varepsilon^k W(\xi) + \cdots$. Then

$$[W, Z^0] = K - Z^k$$

where k contains term which are given or have been previously calculated. Conditions (i) and (ii) are used to prove that the right side of the above is in P_k and condition (iii) gives a solution pair W and Z^k.

For a proof of this theorem see [3] or for a slightly more general theorem see [7]. For more background on Lie brackets see any advanced book on differential geometry, for example [14].

Henceforth, I shall say that an equation of the form (4) is in normal form if $[Z^i, Z^0]=0$ for $i \geqq 0$. This means that the flows generated by Z^0 and Z^i commute.

3. Sternberg's example. In a series of papers, Sternberg [11], [12], [13] considered the problem of constructing a C^k linearization of a differential equation about a critical point or of a diffeomorphism about a fixed point. That is, he asked when does there exist a C^k change of coordinates $x \to y$, valid in a neighborhood of the origin in R^n, which transforms the nonlinear equation

$$\dot{x}=Ax+f(x) \tag{1}$$

into the linear equation

$$\dot{y}=Ay \tag{2}$$

where f is C^k and $f(0)=0$, $\partial f(0)/\partial x=0$. This is just the C^k version of the same question as Poincaré asked a half century before. If $A=\operatorname{diag}(\lambda_1,\cdots,\lambda_n)$, then from the work of Poincaré [8], it follows that $[Ax,f(x)]=0$ (i.e., (1) is in normal form) if and only if f is a sum of terms of the form $e_j x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ where $\lambda_j=\alpha_1\lambda_1+\alpha_2\lambda_2+\cdots+\alpha_n\lambda_n$ and e_j is the vector which has a 1 in position j and zeros elsewhere. Thus, the normal form is nontrivial if there is an integer relation between the eigenvalues.

To obtain this result from the theorem of the previous section, first scale the equation (1) by $x \to \varepsilon x$. Let P_i denote the vector fields on R^n which are homogeneous polynomials of degree $i+1$. A basis for P_i consists of all vectors of the form $e_j x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $j=1,\cdots,n$ and $\alpha_1+\cdots+\alpha_n=i+1$. The operator L_i has these vectors as eigenvectors with eigenvalues $\lambda_j-(\alpha_1\lambda_1+\cdots+\alpha_n\lambda_n)$ as can be shown by a simple computation.

The simplest such relation occurs when there are just 2 eigenvalues, $n=2$, and one eigenvalue is just double the other. Thus, the simplest example with a nontrivial normal form is

$$\dot{\xi}=2\xi+\alpha\eta^2, \qquad \dot{\eta}=\eta. \tag{3}$$

This is the example that Sternberg gives as an equation for which there does not exist a C^2 linearization. (Actually, he gives the analogous example for diffeomorphisms.) There is a C^1 linearization for this equation [4].

When $\alpha=0$, the equations are linear and the general solution is $\xi(t)=e^{2t}\xi_0$, $\eta(t)=e^t\eta_0$. Note that if $\eta_0 \neq 0$, then $\xi(t)=(\xi_0/\eta_0^2)\eta(t)^2$ so all solutions off the ξ-axis lie on parabolas. Thus, as $t \to -\infty$, the solutions approach the origin along analytic curves. Contrast this with the case when $\alpha \neq 0$, say $\alpha=1$. As is typical of an equation in normal form one can still integrate these equations. First solve the η equation and substitute the answer into the ξ equation to get the general solution $\xi(t)=\xi_0 e^{2t}+\eta_0^2 t e^{2t}$, $\eta(t)=\eta_0 e^t$. Now, if $\eta_0 \neq 0$, $\xi(t)=\eta(t)^2\{a+\log|\eta|\}$ where $a=\xi_0/\eta_0^2-\ln|\eta_0|$. This family of curves are C^1, but not C^2 near the origin. Since the property of lying on C^2 curves must be preserved by a C^2 change of coordinates, it is clear that there is no C^2 transformation which linearizes (3) when $\alpha=1$.

Hartman [5] gives the example

$$\dot{\xi}=\alpha\xi, \quad \dot{\eta}=(\alpha-\nu)\eta+\varepsilon\xi\zeta, \quad \dot{\zeta}=-\gamma\zeta$$

where $\alpha>\gamma>0$ and $\varepsilon\neq 0$ as an equation for which there does not exist a C^1 linearization. Note that this equation is in normal form and solvable!

4. Cherry's example. In the second edition of Whittaker's book on dynamics [14], he asserts that the equilateral equilibrium points in the restricted three body problem are stable for small values of the mass ratio parameter. His assertion is based purely on the analysis of the linearized equations. In the third edition this assertion is dropped and an example due to Cherry [2] is included. The restricted three body problem is a conservative dynamical system of two degrees of freedom which depends on a parameter μ, known as the mass ratio parameter. Mathematically, it is defined by a particular Hamiltonian which need not be given here. The system has five equilibrium point for all values of μ. There is a range of values for μ where the linearized equations of motion about one pair of equilibria are similar to two harmonic oscillators. Also, there is one value of μ for which the ratio of the frequencies is 1 : 2.

Cherry's example is given by the Hamiltonian

$$H=\frac{1}{2}\lambda(x_1^2+y_1^2)-\lambda(x_2^2+y_2^2)+\frac{1}{2}\alpha\{x_2(x_1^2-y_1^2)-2x_1y_1y_2\}. \tag{1}$$

The system of differential equations defined by the Hamiltonian (1) is

$$\begin{aligned}
\dot{x}_1&=\frac{\partial H}{\partial y_1}=\lambda y_1-\alpha\{x_2y_1+x_1y_2\},\\
\dot{x}_2&=\frac{\partial H}{\partial y_2}=-2\lambda y_2-\alpha x_1y_1,\\
\dot{y}_1&=-\frac{\partial H}{\partial x_1}=-\lambda x_1-\alpha\{x_1x_2-y_1y_2\},\\
\dot{y}_2&=-\frac{\partial H}{\partial x_2}=2\lambda x_2-(\alpha/2)\{x_1^2-y_1^2\}.
\end{aligned}$$

When $\alpha=0$, these are the equations of two harmonic oscillators with frequencies λ and 2λ. As is well known, a small nonconservative nonlinearity can make a linear oscillator unstable, but this example shows that even a conservative nonlinearity can make the oscillators unstable.

Whittaker gives the solution to the equations whose Hamiltonian is (1). The referee pointed out there are typographical errors in Whittaker's solution and the correct solution is:

$$\begin{aligned}
x_1&=\frac{\sqrt{2}}{\alpha(t+\varepsilon)}\sin(\lambda t+\nu), & y_1&=\frac{\sqrt{2}}{\alpha(t+\varepsilon)}\cos(\lambda t+\nu),\\
x_2&=\frac{1}{\alpha(t+\varepsilon)}\sin 2(\lambda t+\nu), & y_2&=\frac{1}{\alpha(t+\varepsilon)}\cos 2(\lambda t+\nu)
\end{aligned} \tag{2}$$

where ε and ν are integration constants. If we take $\alpha>0$ and $\varepsilon=0$, then these solutions are near the origin for t near minus infinity and arbitrarily large as t approaches 0. Thus the origin is unstable.

Where did this example come from? Since the example is given in rectangular coordinates, it is hard to see that these equations are actually in normal form. When considering a Hamiltonian system associated with harmonic oscillators, it is judicious to use action-angle coordinates. Let $I_i = \frac{1}{2}(x_i^2 + y_i^2)$ and $\phi_i = \arctan(y_i/x_i)$, so that Hamiltonian (1) becomes

$$H = \lambda I_1 - 2\lambda I_2 + \sqrt{2}\,\alpha I_1 I_2^{1/2}\cos(\phi_2 + 2\phi_1). \tag{3}$$

Action-angle variables are canonical or symplectic coordinates. This means that the equations of motion are derived from (3) by the same prescription as used to derive the equations of motion from (1). Thus, the equations of motion in these coordinates are

$$\begin{aligned}
\dot{I}_1 &= \frac{\partial H}{\partial \phi_1} = -2\sqrt{2}\,\alpha I_1 I_2^{1/2}\sin(\phi_2 + 2\phi_1),\\
\dot{I}_2 &= \frac{\partial H}{\partial \phi_2} = -\sqrt{2}\,\alpha I_1 I_2^{1/2}\sin(\phi_2 + 2\phi_1),\\
\dot{\phi}_1 &= -\frac{\partial H}{\partial I_1} = -\lambda - \sqrt{2}\,\alpha I_2^{1/2}\cos(\phi_2 + 2\phi_1),\\
\dot{\phi}_2 &= -\frac{\partial H}{\partial I_2} = 2\lambda - (\sqrt{2}/2)\,\alpha I_2^{-1/2}\cos(\phi_2 + 2\phi_1).
\end{aligned}$$

There is an exact analogue of the theorem in §2 for Hamiltonian systems. One simply replaces the Lie bracket operator $[\cdot,\cdot]$ with the Poisson bracket operator $\{\cdot,\cdot\}$ of Hamiltonian mechanics. The Poisson bracket operator $\{\cdot,\cdot\}$ in these coordinates is defined by

$$\{F, G\} = \sum_{i=1}^{2}\left\{\frac{\partial F}{\partial I_i}\frac{\partial G}{\partial \phi_i} - \frac{\partial F}{\partial \phi_i}\frac{\partial G}{\partial I_i}\right\}.$$

If we set $H_0 = \lambda I_1 - 2\lambda I_2$ and $H_1 = H - H_0$, then it is a simple computation to verify that $\{H_0, H_1\} = 0$ and therefore the Hamiltonian (1) or (3) is in normal form. The normal form for a Hamiltonian that starts with two harmonic oscillators with frequencies ω_1 and ω_2 contains only terms in I_1 and I_2 when ω_1/ω_2 is irrational. Such a Hamiltonian will be stable. Therefore Cherry, like Sternberg in the previous example, chose the simplest case where the normal form was nontrivial, i.e., when the ratio of the two frequencies is 2. The equations resulting from (3) can be explicitly solved by introducing $\theta = \phi_2 + 2\phi_1$ just as in the previous example.

There is another way to see that the origin is unstable. Let $\lambda = 1$, $\alpha = 1/\sqrt{2}$ and consider the Lyapunov function

$$V = I_1 I_2^{1/2}\sin(\phi_2 + 2\phi_1) \tag{4}$$

whose derivative along the trajectories is

$$\dot{V} = 2I_1 I_2 + \frac{1}{2}I_1^2. \tag{5}$$

$I_1 \equiv 0$ is a plane filled with periodic solutions, but if $I_1 > 0$ then V is positive. By Chetaev's theorem [6] the origin is unstable. This simple Lyapunov argument can be extended to include the case when H contains higher order terms. In this case explicit solutions cannot be found in general.

A closely related example is found in Siegal and Moser's book on celestial mechanics [10, pp. 222–224]. There they give an example of an area preserving mapping of the plane that has an unstable fixed point. The eigenvalues of the linearization of the map at the fixed point are qth roots of unity. Again the example is in the appropriate normal form for area preserving maps.

5. Annihilation of invariant tori. This example and the next illustrate how invariant tori behave after perturbations. In both cases an autonomous system is perturbed by a small periodic forcing term which is chosen so that the equations are in normal form.

This example considers perturbations of a two-dimensional autonomous system which depends on a parameter μ in such a way that $\mu > \mu_0$ the system has two limit cycles (one stable and one unstable), a semi-stable limit cycle for $\mu = \mu_0$, and no limit cycle for $\mu < \mu_0$. The simplest model for the unperturbed system is

$$\dot{\rho} = (\mu - \mu_0) - (\rho - \rho_0)^2, \qquad \dot{\theta} = \omega \tag{1}$$

where ω is a nonzero constant (the frequency). In the above (ρ, θ) are polar coordinates in an annular region around the circle $\rho = \rho_0$. One readily sees that this equation admits two limit cycles, $\rho = \rho_0 \pm \sqrt{(\mu - \mu_0)}$ for $\mu > \mu_0$; one limit cycle $\rho = \rho_0$ for $\mu = \mu_0$; and no limit cycle for $\mu < \mu_0$. The periods of these limit cycles are $2\pi/\omega$.

Now add a small nonautonomous, but 2π-periodic, perturbation to equation (1). If we artifically consider equation (1) as a 2π-periodic system, then the limit cycles discussed above become invariant tori in space-time when one identifies the time coordinate modulo 2π. It is natural to suspect that even after the perturbation is added there is a μ^* close to μ_0 such that for $\mu > \mu^*$ there are two invariant tori, for $\mu = \mu^*$ there is one torus and for $\mu < \mu^*$ there are none. The example given below shows that this conjecture is false at least in the resonance case. Chenciner [1] has considered this problem in much greater detail.

In the resonance case is when $\omega = p/q$ where p and q are relative prime integers. The theorem on normal forms in §2 is quoted for autonomous equations. However, the nonautonomous case is included when one uses the standard trick of introducing a new variable τ and augmenting the equations with $\dot{\tau} = 1$. If one takes the unperturbed equation (i.e., (2.3)) to be $\dot{\rho} = 0$, $\dot{\theta} = p/q$, $\dot{\tau} = 1$, then the perturbed equations are in normal form if they are periodic functions of $(q\theta - pt)$. That is, one can eliminate any other type of perturbation by a normalizing transformation. Instead of considering the most general case, consider only the perturbation given by

$$\begin{aligned} \dot{\rho} &= (\mu - \mu_0) - (\rho - \rho_0)^2 + \varepsilon^2 \alpha \cos(q\theta - pt), \\ \dot{\theta} &= p/q + \varepsilon\beta \sin(q\theta - pt). \end{aligned} \tag{2}$$

In the above α and β are positive constants and the system is $2\pi/p$-periodic. Since we wish to study these equations when ρ is near ρ_0 and μ is near μ_0, so scale by $\varepsilon r = \rho - \rho_0$ and let $\varepsilon^2 \nu = \mu - \mu_0$. Then

$$\begin{aligned} \dot{r} &= \varepsilon\{\nu - r^2 + \alpha \cos(q\theta - pt)\}, \\ \dot{\theta} &= p/q + \varepsilon\beta \sin(q\theta - pt). \end{aligned} \tag{3}$$

In the above equation only $q\theta - pt$ appears and so it is natural to introduce a new angle ψ equal to this combination but this would mix the time variable and the spatial variables. In order to keep the geometry straight, introduce a new angular variable τ

defined modulo 2π, augment the equation (3) with the equation $\dot{\tau}=1$ and replace t by τ on the right-hand side of (3). The variables (r,θ,τ) are now variables in $R^1\times S^1\times S^1 = R^1\times T^2$. Since p and c are relative prime integers, there are integers a and b such that $ap+bq=1$. Make the change of variables

$$\psi=q\theta-p\tau,\qquad \sigma=a\theta+b\tau \tag{4}$$

so that the equations become

$$\begin{aligned}\dot{r}&=\varepsilon\{\nu-r^2+\alpha\cos\psi\},\\ \dot{\psi}&=\varepsilon\beta\sin\psi,\\ \dot{\sigma}&=1/q+\varepsilon a\alpha\sin\psi.\end{aligned} \tag{5}$$

Since the coefficients q,p,a,b in (4) are integers and the determinant of the coefficients in (4) is $+1$, the transformation (4) and its inverse preserve the integer lattice Z^2 in R^2. Thus (4) represents a valid change of variables on T^2 or both ψ and σ are angular coordinates defined $\bmod 2\pi$.

Since the first two equations in (5) do not depend on σ, they can be analyzed separately. Also the first two equations in (5) are autonomous so the classical phase plane analysis method can be used. However, (r,ψ) are not polar coordinates in the plane since $r=0$ does not correspond to a point. They should be considered as coordinates on a annulus or a cylinder. In our figures one should identify points $\bmod 2\pi$ in ψ.

These equations have critical points at $\psi=0$, $r=\pm\sqrt{\nu+\alpha}$ and $\psi=\pi$, $r=\pm\sqrt{\nu-\alpha}$. Linearizing about these critical points gives that the critical points are of the following types:

$\psi=0$, $r=+\sqrt{\nu+\alpha}$ is a saddle,
$\psi=0$, $r=-\sqrt{\nu+\alpha}$ is a source,
$\psi=\pi$, $r=+\sqrt{\nu-\alpha}$ is a sink,
$\psi=\pi$, $r=-\sqrt{\nu-\alpha}$ is a saddle.

In the above ν is assumed large enough that the square roots are real and nonzero. The critical points along $\psi=0$ undergo a saddle-node bifurcation when $\nu=-\alpha$, whereas the critical points along $\psi=\pi$ undergo a saddle-node bifurcation when $\nu=+\alpha$. Also note that the lines $\psi=0$ and π are invariant and that the flows along these lines are as shown in Fig. 1a when $\nu>+\alpha$. When referring to Figs. 1a, b, c, recall that $r=0$ corresponds to the circle $\rho=\rho_0$.

When $\nu>+\alpha$ all four critical points exist, r is decreasing on $r=\pm 2\sqrt{\nu+\alpha}$, and r is increasing on $r=0$. Consider the half of the unstable manifold of the saddle point at $\psi=0$, $r=+\sqrt{\nu+\alpha}$ which lies in the upper half plane. It is trapped in the region $0\leqq r\leqq 2\sqrt{\nu+\alpha}$, $0\leqq\psi\leqq\pi$. There are no critical points in the interior of this region and ψ is increasing. Thus this unstable manifold must approach the sink at $\psi=\pi$, $r=+\sqrt{\nu-\alpha}$ as $t\to\infty$. The same is true for the other half of the unstable manifold of the critical point at $\psi=0$, $r=+\sqrt{\nu+\alpha}$. Thus the first two equations in (5) have an invariant circle for $\nu>+\alpha$ which consists of the saddle point at $\psi=0$, $r=+\sqrt{\nu+\alpha}$ and its unstable manifold plus the sink at $\psi=\pi$, $r=+\sqrt{\nu-\alpha}$. A similar analysis (by reversing time) shows that when $\nu>+\alpha$ there is another invariant circle consisting of the saddle $\psi=\pi$, $r=-\sqrt{\nu-\alpha}$ and its stable manifold plus the source at $\psi=0$, $r=$

$-\sqrt{\nu+\alpha}$. See Fig. 1a. These invariant curves are smooth except possibly at the source and sink where they may have a kink.

The same type of analysis can be carried out when $\nu=+\alpha$ to show that these first two equations in (5a) have an invariant set which consists of two circles which meet at $\psi=\pi$, $r=0$ as shown in Fig. 1b. A similar analysis yields Fig. 1c when $\alpha>\nu>-\alpha$.

I claim that Figs. 1a, b, c also represent a picture of the section map for the full set of equations in (5) where the surface of a section is taken as $\sigma=0$. Let $r(t,r_0,\psi_0)$, $\psi(t,r_0,\psi_0)$, $\sigma(t,r_0,\psi_0)$ be the solution of equations (5) which pass through $r=r_0$, $\psi=\psi_0$, $\sigma=0$ when $t=0$. Clearly $\sigma(t,r_0,\psi_0)=t/q+O(\varepsilon)$ so we may apply the implicit function theorem to the equation $\sigma(t,r_0,\psi_0)=2\pi$ to yield the existence of a smooth function $T(r_0,\psi_0)=2\pi q+O(\varepsilon)$ which is the first return time to the $\sigma=0$ section. The section map is then the map $(r_0,\psi_0)\rightarrow(r(T(r_0,\psi_0),r_0,\psi_0),\psi(T(r_0,\psi_0),r_0,\psi_0))$. Thus the section map is obtained by following the flow of the first two equations in (5) by a time $T=2\pi q+O(\varepsilon)$.

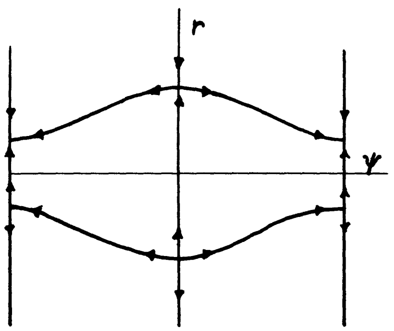

FIG. 1a

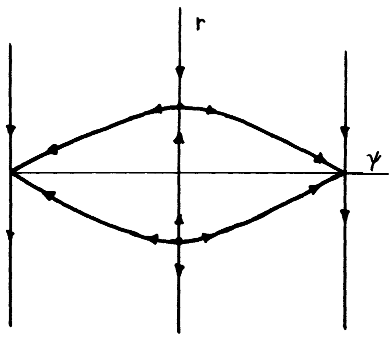

FIG. 1b

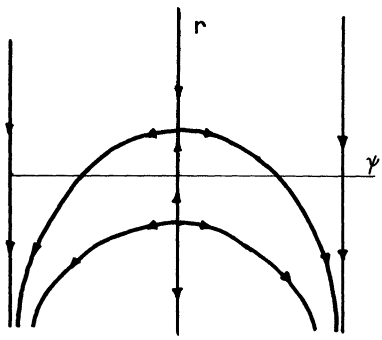

FIG. 1c

Going back to the full three-dimensional problem, one sees that the equation (5) and hence equations (3) and (2) have two invariant tori when $\mu>+\alpha$ which contain two periodic solutions per torus. As μ approaches $+\alpha$ from above, these tori approach each other along the circle $\psi=\pi$, $r=0$, σ arbitrary and at $\mu=+\alpha$ the equations admit an invariant set which consists of two tori which have this circle in common. When $+\alpha>\mu>-\alpha$, the equations still have two periodic solutions but their unstable manifolds no longer form tori. As $\mu\rightarrow-\alpha$ from above the two remaining periodic solutions undergo a saddle-node bifurcation and disappear.

Remark. Figures 1a, b, c do not represent the period map for equations (2) or (3). The surface $\sigma \equiv 0 \bmod 2\pi$ corresponds to $a\theta + b\tau \equiv 0 \bmod 2\pi$ in the original variables. The picture of the period map can be obtained from these pictures by linearly contracting any of these figures in the ψ direction until it has width $2\pi/q$. Now fill out the figure by duplicating the drawing q times in $j2\pi \leqq \theta < (j+1)2\pi$, $j=0,\cdots,q-1$.

6. Hopf bifurcation for invariant tori. This example was suggested by Professor George Sell and should be considered as an example which proves the necessity of the resonance condition in his paper [9]. His paper proves a theorem which is a natural extension of Hopf's bifurcation theorem to invariant tori. The precise statement of the theorem is rather lengthy and is not needed for the discussion given below. The main hypothesis is the existence of a one-parameter family of invariant n-tori with certain conditions on the linear variational equations which are analogous to the derivative conditions of Hopf's original theorem. The main conclusion is that for a certain value of the parameter an $(n+1)$-torus bifurcates from the n-torus. Sell found it necessary to assume that at the point of bifurcation the flow on the n-torus is a linear ergodic (irrational) flow.

The example is a perturbation of an autonomous three-dimensional system which depends on a parameter μ. The unperturbed autonomous system has a limit cycle for all values of the parameter μ, but for $\mu=0$ it undergoes a Hopf bifurcation, i.e. an invariant two torus bifurcates from the limit cycle as the limit cycle changes stability. The unperturbed equation is

$$\dot{r}=\mu r-r^3, \quad \dot{\theta}=\omega, \quad \dot{\phi}=\lambda. \tag{1}$$

Here (r,θ,ϕ) are coordinates for a solid torus, i.e., (r,θ) are polar coordinates in R^2 and ϕ is a coordinate on S^1 (see Fig. 2c). When written in rectangular coordinates, the first two equations in (1) are analytic at the origin. Both θ and ϕ are angular variables defined modulo 2π and $r \geqq 0$. The equations (1) admit a limit cycle $r=0$ for all values of μ which is asymptotically stable when $\mu<0$ and unstable when $\mu>0$. For $\mu>0$ this limit cycle is encircled by a stable invariant torus $r=\sqrt{\mu}$, θ and ϕ arbitrary and the flow on this invariant torus is the linear flow $\dot{\theta}=\omega$, $\dot{\phi}=\lambda$.

Add to (1) a small nonautonomous, but periodic, perturbation which is in resonance with one of the frequencies, say λ. If (1) is artificially considered as a 2π-periodic system, the limit cycle becomes an invariant three torus and the invariant two torus becomes an invariant three torus in the space-time (r,θ,ϕ,t) where t is an angular variable modulo 2π. As in the previous examples, the perturbation terms are chosen as the simplest nontrivial terms as predicted by the theory of normal forms.

Let the perturbed equations be

$$\begin{aligned} &\dot{r}=\mu r-r^3+\varepsilon^2\cos(q\phi-pt), \\ &\dot{\theta}=\omega+\varepsilon^2 f(q\phi-pt), \\ &\dot{\phi}=p/q+\varepsilon^2\sin(q\phi-pt) \end{aligned} \tag{2}$$

where f is an arbitrary, smooth, 2π-periodic function whose precise form is unimportant. In order to investigate what happens when μ and r are small, scale by $r \to \varepsilon r$ and let $\mu \to \varepsilon^2\mu$. As before introduce τ and change coordinates by $\psi=q\phi-p\tau$, $\sigma=a\phi+b\tau$ where $qb+ap=1$. The equations become

$$\dot r = \varepsilon^2\{\mu r - r^3 + r\cos\psi\},$$
$$(3) \qquad \dot\theta = \omega + \varepsilon^2 f(\psi),$$
$$\dot\psi = \varepsilon^2 q \sin\psi,$$
$$\dot\sigma = 1/q + \varepsilon^2 a \sin\psi.$$

For the moment ignore the θ and σ equations and call the remaining equations for r and ψ equations (3′). The coordinates (r,ψ) are not polar coordinates in the plane since $r=0$ does not correspond to a point. However, $r \geqq 0$. These equations have two critical points: $r=\sqrt{\mu+1}$, $\psi=0$; and $r=\sqrt{\mu-1}$, $\psi=\pi$. When $\mu>-1$ the critical point at $r=\sqrt{\mu+1}$, $\psi=0$ is a saddle and when $\mu>1$ the critical point at $r=\sqrt{\mu-1}$, $\psi=\pi$ is a sink. When $\mu>+1$ both critical points exist, r is decreasing for large r and r is increasing for small r. Thus the unstable manifold of the saddle must approach the sink at $r=\sqrt{\mu-1}$, $\psi=\pi$ as shown in Fig. 2a.

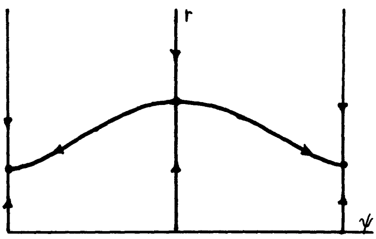

FIG. 2a

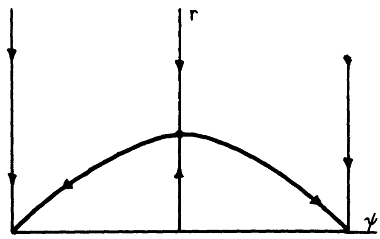

FIG. 2b

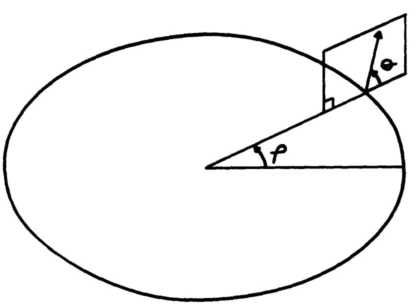

FIG. 2c

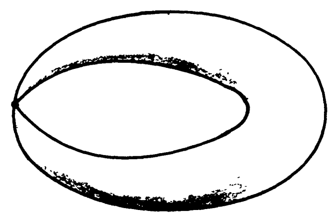

FIG. 2d

As $\mu \to +1$ from above, the sink approaches the point where $r=0$ and at $\mu=1$ only the saddle persist. For all μ, r is decreasing for r large and so for $1 \geqq \mu > -1$ the unstable manifold of the saddle approaches a point where $r=0$ as shown in Fig. 2b. As $\mu \to -1$ from above the saddle approaches the set where $r=0$ and for $\mu \leqq -1$ there are not critical points.

Thus for $\mu>+1$ equations (3′) have an important circle which is made up of the unstable manifold of the saddle and the sink and which is smooth except possibly at the sink. As $\mu \to +1$ from above the invariant circle approaches the point $r=0$, $\psi=\pi$ and for $\mu=+1$ the invariant circle attaches to the point where $r=0$, $\psi=\pi$. For $1 \geqq \mu > -1$ the invariant circle is attached to the set $r=0$ but as $\mu \to -1$ the invariant circle approaches the set $r=0$.

Now return to the full set of equations in (3). If ε is sufficiently small θ and σ are always increasing since $\omega > 0$, $1/q > 0$. Recall that (r, θ) are polar coordinates in R^2 while ψ and σ are angular variables in S^1. Thus $r = 0$ corresponds to a two-dimensional torus. Since $r = 0$ is invariant, equations (3) always admit an invariant two torus. But for $\mu > +1$ the equations (3′) also admit an additional invariant circle and so equations (3) admit an invariant three torus. As $\mu \to +1$ from above the invariant three torus approaches the invariant two torus along the $\psi = \pi$ direction. For $1 \geqq \mu > -1$ the invariant two torus remains, but now the other invariant set is a pinched three torus (pinched along an S^1). To see this, consider the unstable manifold of the saddle at $\psi = 0$, $r = \sqrt{\mu + 1}$ (see Fig. 2b) plus the saddle point itself. Above each point of this curve there is a two torus, θ and σ arbitrary, but as you approach the critical point at the right in Fig. 2b, r tends to zero. Thus above the point $r = 0$ there is a single circle, σ arbitrary. The two torus and the pinched three torus have a circle in common, namely $r = 0$, $\psi = \pi$ and σ arbitrary. As $\mu \to -1$ the pinched three torus approaches the two torus and disappears. For $\mu < -1$ the two torus remains as a sink.

In order to picture the above bifurcation, consider the $\sigma = 0$ section map. This map will have the circle $r = 0$ as invariant. For $-1 > \mu > 1$ this section map will have an invariant set which is a pinched two torus as shown in Fig. 2d.

It is important to note that equations (2) are analytic in the rectangular coordinates corresponding to the polar coordinates (r, θ).

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