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BIFURCATIONS OF RELATIVE EQUILIBRIA IN THE N-BODY AND KIRCHHOFF PROBLEMS*

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Abstract. The bifurcations of a one-parameter family of relative equilibria in the N-body problem are studied using normal form theory, Lie transforms, and an algebraic processor. The one-parameter family consists of $N-1$ bodies of mass 1 at the vertices of a regular polygon and one body of mass m at the centroid. As N increases there are more and more values of the mass parameter m where the relative equilibrium is degenerate. For $N \leqq 13$ each of these degenerates gives rise to a bifurcation and a new relative equilibrium. This is established using a computer-aided proof. A similar analysis is carried out for the N-vortex problem of Kirchhoff.

Key words. central configurations, relative equilibria, N-body, bifurcation

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1. Introduction. The study of relative equilibria (r.e.) of the N-body problem has had a long history starting with the famous collinear configuration of the 3-body problem found by Euler (1767). Over the intervening years many different technologies have been applied to the study of r.e. In the older papers of Euler (1767), Lagrange (1772), Hoppe (1879), Lehmann-Filhes (1891), and Moulton (1910), special coordinates, symmetries, and analytic techniques were used. In their investigations, Dziobek (1900) used the theory of determinants; Smale (1970) used Morse theory; Palmore (1975) used homology theory; Simo (1977) used a computer; and Moeckel (1985) used real algebraic geometry. Thus, the study of r.e. has been a testing ground for many different methodologies of mathematics.

In Meyer and Schmidt (1987) the methods of bifurcation analysis and the use of the automated algebraic processor were brought to bear on this subject and the present paper continues the attack. Specifically we study the bifurcations of the relative equilibrium which consists of $N-1$ particles of mass 1 at the vertices of a regular polygon and one particle of mass m at the centroid. We call this the regular polygon relative equilibrium (r.p.r.e.). Our first paper considers the 4- and 5-body problems and uses the special coordinates of Dziobek (1900). These coordinates make the 4-body problem relatively easy to handle and the 5-body problem accessible, but beyond 5, Dziobek's coordinates become very cumbersome. The 4- and 5-body problems in these special coordinates are sufficiently simple that the general purpose algebraic processor MACSYMA could handle the tedious calculations. For larger N the special purpose algebraic processor POLYPAK written by the second author was needed because the computations increased rapidly with N. In the analysis of the 4- and 5-body problems the classical power series methods of bifurcation analysis handles the problems nicely, but for larger n a systematic use of Lie transforms by Deprit (1969) was mandated in order to bring the equations into a normal form. Thus this paper uses substantially different techniques than our previous paper.

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The problem of finding an r.e. can be reduced to finding a critical point of the potential energy function on the manifold of constant moment of inertia. Thus the problem falls within the domain of catastrophe theory and so the general theory is well understood. However, this specific problem has a high degree of symmetry, many variables, and a constraint, so the computations must be performed with care. We consider this paper as a case study in bifurcation analysis in the face of these complexities.

Indeed we analyze the problem at three different computational levels. First, for small N, we perform the normalization to high order to determine the existence, uniqueness, and exact shape of the bifurcating equilibria. For medium ranges of N we exploit the symmetry so that fewer computations need be carried out in order to establish existence, but now the uniqueness is only within the class of symmetric equilibria. Last, for large N, we carry out some calculations to establish existence of bifurcations with no uniqueness information. We can see that for a fixed amount of computing power the precision of the information obtained decreases as N increases.

For the planar problem that we consider, a relative equilibrium is also a central configuration and vice versa, that is, a homothetic solution which begins or ends in total collapse or tends to infinity. Even though as solutions of the N-body problem r.e. are quite rare and rather special, they are of central importance in the analysis of the asymptotic behavior of the universe. In general, solutions which expand beyond bounds or collapse in a collision do so asymptotically to a central configuration. A survey and entrance to this literature can be found in Saari (1980).

Interestingly this problem in celestial mechanics is formally similar to the problem in fluid dynamics of describing the evolution of finitely many interacting point vortices in the plane. Kirchhoff (1897) shows that this problem is specified by a Hamiltonian which is similar to the Hamiltonian of an N-body problem with a logarithmic potential. The constants that correspond to the masses are now the circulations, which may be positive or negative, and so a richer store of bifurcations are to be expected. We develop the theory and evolution of the bifurcations of the problem in parallel with that of the N-body problem.

In Meyer and Schmidt (1987) we studied the 4- and 5-body problem and found that there was a unique value of the mass of the central particle where the potential was degenerate. This agrees with the findings in Palmore (1973). However, for larger N there are more and more values of this mass at which the potential is degenerate, which disagrees with Palmore (1976). In fact, for large N many bifurcations occur. We developed the general theory of the bifurcations for these two problems for all N and completely analyze the bifurcations for $4 \leqq N \leqq 13$. Figures 1 and 2 illustrate the bifurcations which occur at the unique critical mass when $N = 4, 5$ and Fig. 3 illustrates the multitude of bifurcations that occurs in the 13-body problem.

Also we found that the self-potential for the N-body problem with the central mass removed was not always a nondegenerate minimum. In fact it is a saddle for $N > 6$. This disagrees with one of the findings in Palmore (1975). There were other surprises in our investigations, which will be explained below when we have developed the necessary definitions and notation.

2. Relative equilibria for the N-body and Kirchhoff problems. The N-body problem is the system of differential equations that describes the motion of N particles moving under the influence of their mutual gravitational attraction. Let $q_j \in R^2$ be the position vector, $p_j \in R^2$ the momentum vector, and $m_j > 0$ the mass of the jth particle, $1 \leqq j \leqq N$;

then the equations of motion are

$$\dot{q}_j = \frac{\partial H}{\partial p_j} = \frac{1}{m_j} p_j, \qquad \dot{p}_j = -\frac{\partial H}{\partial q_j} = \frac{\partial U}{\partial q_j}, \qquad j = 1, \cdots, N \tag{2.1}$$

where H is the Hamiltonian

$$H = \sum_{j=1}^{N} \frac{\|p_j\|^2}{2m_j} - U(q) \tag{2.2}$$

and U is the (self) potential

$$U = \sum_{1 \leqq i < j \leqq N} \frac{m_i m_j}{\|q_i - q_j\|}. \tag{2.3}$$

These equations reduce to the Newtonian formulation

$$m_j \ddot{q}_j = \frac{\partial U}{\partial q_j}, \qquad j = 1, \cdots, N. \tag{2.4}$$

To change to rotating coordinates let $q_j = \exp(\nu J t) u_j$ where $\nu > 0$ is the frequency of the rotating frame and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

so (2.4) becomes

$$m_j\{\ddot{u}_j + 2\nu J \dot{u}_j - \nu^2 u_j\} = \frac{\partial U}{\partial u_j}(u), \qquad j = 1, \cdots, N. \tag{2.5}$$

An equilibrium in these rotating coordinates is a solution of the system of algebraic equations

$$-\lambda m_j u_j = \frac{\partial U}{\partial u_j}, \qquad j = 1, \cdots, N \tag{2.6}$$

where $\lambda = \nu^2 > 0$.

The Kirchhoff problem is the system of differential equations describing the motion of N vortices moving in the plane under their mutual interaction. Let q_j be the position vector and $m_j \neq 0$ the circulation of the jth vortex for $j = 1, \cdots, N$. Then Kirchhoff (1897) gives the equations of motion as

$$m_j \dot{q}_j = J \frac{\partial U(q)}{\partial q_j}, \qquad j = 1, \cdots, N \tag{2.7}$$

where now U is the Hamiltonian

$$U = -\sum_{1 \leqq i < j \leqq N} m_i m_j \log \|q_i - q_j\|. \tag{2.8}$$

Introducing rotating coordinates as before by setting $q_j = \exp(\nu Jt)u_j$ transforms (7) to the system

$$m_j\{\dot{u}_j + \nu J u_j\} = J\frac{\partial U(u)}{\partial u_j}, \qquad j = 1, \cdots, N. \tag{2.9}$$

An equilibrium in these rotating coordinates is a solution of the system of algebraic equations

$$-\lambda m_j u_j = \frac{\partial U(u)}{\partial u_j}, \qquad j = 1, \cdots, N \tag{2.10}$$

where $\lambda = -\nu$.

It is classical and easy to verify that if $\bar{u} = (\bar{u}_1, \cdots, \bar{u}_N)$ and $\bar{\lambda}$ is a solution of (6) (or (10), respectively), then the center of mass of $\bar{u}$ is at the origin ($\sum m_j \bar{u}_j = 0$) and $\bar{\lambda} = U(\bar{u})/(1+\delta)I(\bar{u})$ (>0 for (6)) where I is the moment of inertia

$$I(u) = \tfrac{1}{2}\sum m_j \|u_j\|^2 \tag{2.11}$$

and $\delta = 0$ for the Kirchhoff problem or $\delta = 1$ for the N-body problem. For either problem we will set

$$\begin{aligned} M &= \{u \in R^{2N} : \textstyle\sum m_j u_j = 0\}, \\ \Delta &= \{u \in R^{2N} : u_i = u_j \text{ for some } i \neq j\}, \\ S &= \{u \in M : I(u) = 1\}. \end{aligned} \tag{2.12}$$

The variable λ can be considered a Lagrange multiplier and so an equivalent definition of a relative equilibrium is a critical point of U restricted to $S \backslash \Delta$. If u is an r.e. then so is $Au = (Au_1, \cdots, Au_N)$ where $A \in \mathrm{SO}(2, R)$ is a rotation matrix. We can define an equivalence relation by $u \sim Au$ when $A \in \mathrm{SO}(2, R)$, and since U, I, are constant on equivalence classes we can define the quotient spaces $\mathcal{S} = (S \backslash \Delta)/\sim$ and the function $\mathcal{U} : \mathcal{S} \to R$ by $\mathcal{U}([u]) = U(u)$, where $[\ ]$ denotes an equivalence class. $\mathcal{S}$ and $\mathcal{U}$ are smooth. Thus a similarity class of r.e. is a critical point of $\mathcal{U}$.

A relative equilibrium is called nondegenerate if its equivalence class is a nondegenerate critical point of $\mathcal{U}$ in the sense of Morse theory, i.e., the Hessian is nonsingular at the critical point. It follows from the implicit function theorem that bifurcations can occur only at degenerate critical points, so first we must find degenerate r.e.

3. Palmore coordinates. Our first step is to introduce the local coordinate system on the quotient space $\mathcal{S}$ which was given in Palmore (1976). Let $n = N - 1$ and $\omega = \exp(i2\pi/n)$ be a primitive nth root of unity. Consider complex numbers as vectors in the plane, so ω^j, $0 \leqq j < n$ are the vertices of a regular polygon with n sides. By the regular polygon relative equilibrium (r.p.r.e.) we shall mean the r.e. which consists of n particles of unit mass, $m_j = 1$, situated at ω^j for $j = 0, \cdots, n-1$, and one particle of arbitrary mass, $m_n = m$, situated at the origin.

Let $q = (q_0, q_1, \cdots, q_n)^T$ be the position vector of the $N = n + 1$ particles in the plane, $\Omega = (\omega^0, \omega^1, \omega^2, \cdots, \omega^{n-1}, 0)^T$ be the position vector of the r.p.r.e., and change coordinates by

$$q = \Omega + \mathcal{V}z \tag{3.1}$$

where $z=(z_0, z_1, \cdots, z_n)$ is the position vector in the new Palmore coordinates and $\mathcal{V}$ is the matrix

$$\mathcal{V}=\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & \omega^1 & \omega^2 & \cdots & \omega^{n-1} & 1 \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2} & 1 \\ 0 & 0 & 0 & \cdots & 0 & -n/m \end{pmatrix}. \tag{3.2}$$

In these coordinates the center of mass is

$$\sum_{j=0}^{n} m_j q_j = n z_0 \tag{3.3}$$

so setting $z_0=0$ fixes the center of mass at the origin. The moment of inertia is

$$\begin{aligned} I &= \frac{1}{2}\sum_{j=0}^{n-1} \|q_j\|^2 + \frac{m}{2}\|q_n\|^2 \\ &= \frac{1}{2}\left(n + z_1 + \bar{z}_1 + \sum_{j=1}^{n} z_j \bar{z}_j + \frac{n}{m}\|z_n\|^2\right) \end{aligned} \tag{3.4}$$

so that the first approximation, the manifold $I=I_0=n/2$, is given by $z_1+\bar{z}_1 = 2 \operatorname{Re} z_1 = 0$. Requiring z_1 to be real, $\operatorname{Im} z_1 = 0$, we select a representative from the rotational equivalence class. Thus to the first approximation local coordinates on $\mathcal{S}$ near $[\Omega]$ are $z_0=z_1=0$ and $z_2, z_3, \cdots, z_n$ arbitrary.

Henceforth, set $z_0=0$ and $\operatorname{Im} z_1=0$ and let $\operatorname{Re} z_1 = x_1$. From (3.4) we see that $\partial I/\partial x_1(\Omega) = 1 \neq 0$, so by the implicit function theorem we can solve $I=I_0$ for x_1, as a function of the remaining variables. Let $x_1=\phi(z_2, z_3, \cdots, z_n)$ be this solution. Changing variables by $x_1' = x_1 - \phi(z_2, \cdots, z_n)$, $z_2'=z_2, \cdots, z_n'=z_n$ brings the manifold $I=I_0$ to the hyperplane $x_1=0$ locally. Thus $z_2', \cdots, z_n'$ are valid local coordinates on $\mathcal{S}$ near $[\Omega]$.

Computationally we effect this change of variables by using the method of Lie transforms as given by Deprit (1969). We construct the change of variables from the unprimed to the primed variables order by order using the standard normalization procedure. That is, we eliminate the x_1 dependence in I order by order. Henceforth, we will assume that this initial normalization has been carried out, we will ignore z_0 and z_1, and we will drop the primes on the variables.

The next step is to look at the Hessian of the function $\mathcal{U}$ at $[\Omega]$. We can consider (1) as a change to the $(z, \bar{z})$ coordinates or follow Palmore and use the real and imaginary parts of z. We choose the latter for exposition purposes.

Let $z_j = x_j + iy_j$, $\xi = (x_2, \cdots, x_n)^T$, $\eta = (y_2, \cdots, y_n)^T$, $u=(x_2, \cdots, y_n)^T$, and $A = \partial^2\mathcal{U}/\partial u^2[\Omega]$. Palmore (1976) shows that the Hessian, A, has the relatively simple form

$$A = \begin{pmatrix} B+C & 0 \\ 0 & B-C \end{pmatrix} \tag{3.5}$$

where B and C are $(n-1)\times(n-1)$ matrices, B is a standard diagonal matrix, and C has nonzero entries only on the cross diagonal running northeast. These nonzero entries are given in Appendix A for reference. In Appendix A we give the general formulas for all potentials which vary inversely with the distance of the δ power, so the N-body problem is when $\delta=1$, and the Kirchhoff problem is the limiting case when $\delta=0$.

Let $D(n, k)$ (respectively, $D^{-}(n, k)$), $2 \leqq k \leqq n/2$ be the 2×2 submatrix of $B+C$ (respectively, $B-C$) formed by taking the (k, k), $(k, n+2-k)$, $(n+2-k, k)$, and $(n+2-k, n+2-k)$ entries. In the case that n is even the two diagonals B and C cross in a single entry at the $(n/2+1, n/2+1)$ position; let $D(n, n/2+1)$ (respectively, $D^{-}(n, n/2+1)$) be the corresponding 1×1 matrix or number. This is a special case, which requires special treatment, and we will typically discuss this case last.

The r.p.r.e. is degenerate when A is singular, which happens when one of the submatrices $D(n, k)$ is singular. Except for the last row of $B \pm C$ all the nonzero entries of $B \pm C$ are linear in m and in fact the determinants of the $D(n, k)$, $2 < k \leqq n/2+1$, are linear in m also. Referring to Appendix A shows that the last row is slightly more complicated, the determinant $D(n, 2)$ has an extraneous factor of $(m+n)$ and another linear factor in m. Thus, there is a unique $m = m(n, k)$, which makes the submatrices $D(n, k)$ and $D^{-}(n, k)$ singular. In the special case when n is even and $k = n/2+1$ the 1×1 matrix or number $D^{-}(n, k)$ does not contain m, and so when $m = m(n, k)$ we have $D(n, k) = 0$ but $D^{-}(n, k) \neq 0$. In this special case the dimension of the kernel of A is one. Let $d(n, k) = \det D(n, k)$ for $m = 0$, $2 < k \leqq n/2+1$. Appendix A also contains the general formulas for $m(n, k)$ and Appendix B contains a table of $m(n, k)$ and $d(n, k)$ for all $3 \leqq n \leqq 12$ for both the N-body problem and the Kirchhoff problem. Recall that $d(n, 2)$ is not defined. The tables in Appendix B are easily generated from the formulas in Appendix A.

Palmore (1973) considered this one-parameter family of r.e. for the $N = n-1$ body problem for $n = 3$, 4 and showed that there was a unique positive value of the mass that makes this r.e. degenerate. In Meyer and Schmidt (1987), we verify this fact and show that additional families of r.e. bifurcate from the original family. Palmore (1976) makes a similar statement about the existence of a unique positive critical mass for all n. From the table in Appendix B, we see that $m(6, 2) \simeq 20.91$ and $m(6, 4) \simeq .00598$, and so this is not the case for $n = 6$. We computed this table all the way up to $n = 20$ and found that as n increases, more and more positive critical masses appear. Moreover, the critical mass of Palmore is $m(n, 2)$ in our notation, and it becomes negative at $n = 7$ and remains negative up to $n = 20$. Later we will show that these positive critical masses give rise to new families of r.e. which bifurcate from the r.p.r.e.

Palmore (1982) also states that there is a unique positive circulation which makes the Kirchhoff potential degenerate for all $n \geqq 3$. Appendix B shows that the uniqueness is false for $8 \leqq n \leqq 12$ and we extended the table to $n = 20$ to find more and more positive critical circulations as n increases. For the Kirchhoff problem the exact formula for the critical circulation $m(n, k)$ takes on a simple form as shown in Appendix A. From this we see that $m(7, 4) \equiv 0$, so one critical circulation is zero. Negative values of the circulation are meaningful and so we investigate these bifurcations in the next section also.

There are several other errors in Palmore (1976), (1982). He also states that the r.e. when $m = 0$ is a nondegenerate maximum of the potential for both problems and for all n. Since we work with the self-potential or the negative of the potential, this would mean that the matrices obtain by deleting the first and last rows and columns of $B \pm C$ are positive definite and in particular the $d(n, k) > 0$ for $2 < k \leqq n/2+1$. This is false when $6 \leqq n \leqq 12$, which can be seen easily by looking at determinants $d(n, k)$ given in Appendix B—in particular $d(6, 4) \simeq -0.036 < 0$. Again we extended this all the way to $n = 20$. We give a simple analytic argument in Appendix D which shows that the potential does not have a minimum at this r.e. when $n = 6$. As noted above the Kirchhoff problem is degenerate when $n = 7$ since $m(7, 4) \equiv 0$. The source of all these errors seems to be in the analysis of the 2×2 submatrices $D(n, k)$.

4. The splitting lemma, reflections, and Hopf's method. There is a simple argument due to Hopf (1942) that establishes a bifurcation without a knowledge of higher-order terms. The analysis of the Hessian given in the previous section along with the symmetry of the potential function is enough to adapt Hopf's argument to the present situation. We present this argument before the discussion of the full normalization to emphasize how little computation is necessary to establish some information about the nature of the bifurcation.

Fix n and k, let $\mu = m - m(n, k)$ and $h = 2n - 2$. The special case when n is even and $k = n/2 + 1$ will be treated at the end, so for now assume we are not in this case. By the analysis of the previous section and the splitting lemma as found in Poston and Stewart (1978) there is a coordinate system η near $[\Omega]$ so that

$$\mathcal{U} = \pm\eta_3^2 \pm \eta_4^2 \pm \cdots \pm \eta_h^2 + G(\eta_1, \eta_2, \mu). \tag{4.1}$$

In catastrophe theory the Lyapunov-Schmidt method is called the splitting lemma. In the next section we discuss in detail how the quadratic terms are brought into the above form and how the function G is computed order by order using Deprit's method of Lie transforms and the second author's algebraic processor POLYPAK.

From the form of the Hessian A in (3.5) and the fact that the submatrices $D(n, k)$ and $D^-(n, k)$ have the same determinant which is linear in the mass m, we see that the quadratic terms of G have the form $\alpha\mu(\eta_1^2 + \eta_2^2)/2$ where α is a nonzero constant.

Also $\mathcal{U}$ is invariant under a reflection $\mathcal{R}$ which leaves the regular polygon relative equilibrium fixed. In the original coordinates the reflection is

$$\mathcal{R} : q_j \to \bar{q}_{n-j}, \quad 0 \leqq j < n, \qquad q_n \to \bar{q}_n. \tag{4.2}$$

At one of the critical masses a perturbation in the direction of the kernel of the Hessian is of the form

$$\begin{aligned} q_j &= \omega^j + \omega^{jk} z_k + \omega^{jl} z_l, \qquad k + l = n + 2, \\ q_j &= \omega^j + \omega^{jk} z, \qquad k = l = n/2 + 1. \end{aligned} \tag{4.3}$$

In the first case the z_k and z_l are not independent but are linearly related (essentially conjugates), so one can be used as a coordinate of the perturbation. In the second case the z is arbitrary. Thus we can use z_k or z as a coordinate in the kernel of the Hessian. The action of $\mathcal{R}$ on this subspace is

$$\begin{aligned} &\mathcal{R} : \omega^j + \omega^{jk} z_k + \omega^{jl} z_l \to \omega^j + \omega^{jk} \bar{z}_k + \omega^{jl} \bar{z}_l, \qquad k \neq l, \\ &\mathcal{R} : \omega^j + \omega^{jk} z \to \omega^j + \omega^{jk} \bar{z}, \qquad k = l = n/2 + 1. \end{aligned} \tag{4.4}$$

Thus in coordinates $\mathcal{R} : z_k \to \bar{z}_k$ or $\mathcal{R} : z \to \bar{z}$, so $\mathcal{R}$ is a reflection on this subspace also. Therefore, we can choose the coordinates η_1 and η_2 so that

$$G(\eta_1, \eta_2, \mu) \equiv G(\eta_1, -\eta_2, \mu). \tag{4.5}$$

This is essentially the same as Lyapunov-Schmidt reduction in the presence of symmetry discussed in Proposition 3.3 of Golubitsky and Schaeffer (1985).

Thus, if $\partial G/\partial\eta_1(\bar{\eta}_1, 0, 0) = 0$, then $\eta = (\bar{\eta}_1, 0, \cdots, 0)$ is a critical point of $\mathcal{U}$. Since $\partial G/\partial\eta_1(0, 0, \mu) = 0$, η_1 is a factor of $\partial G/\partial\eta_1(\eta_1, 0, \mu)$, and so we must solve

$$\begin{aligned} \frac{\partial G}{\partial \eta_1}(\eta_1, 0, \mu) &= \mu\alpha\eta_1 + \eta_1 g(\eta_1, \mu) \\ &= \eta_1(\alpha\mu + g(\eta_1, \mu)) \end{aligned} \tag{4.6}$$

or

$$\alpha\mu + g(\eta_1, \mu) = 0 \tag{4.7}$$

where $g(0, \mu) \equiv 0$. Since $\alpha \neq 0$, the implicit function theorem gives a solution of (3) of the form $\mu = \upsilon(\eta_1)$ and so $\eta = (\eta_1, 0, \cdots, 0)$ is a critical point of $\mathcal{U}$ when $\mu = \upsilon(\eta_1)$. So we have shown that locally the critical point set of $\mathcal{U}$ in $R^h \times R^1$ consists of two intersecting curves namely $(\eta, \mu) \equiv (0, \mu)$ and $(\eta, \mu) = ((\eta_1, 0, \cdots, 0), \upsilon(\eta_1))$. These solutions are symmetric with respect to the reflection $\mathcal{R}$ and are unique in this class. Of course there may be more nonsymmetric solutions.

Hopf's argument just given depends only on the analysis of the Hessian and the symmetry of the system and so is quite easy to apply. The values of $m(n, k)$ and the corresponding α's are easy to compute from the formulas in Appendix A for both the N-body problem and the Kirchhoff problem. Appendix B contains a table of $m(n, k)$ and Appendix C a table of α for $3 \leqq n \leqq 12$. Since the computed values of the α's are nonzero the above result holds in all these cases.

However, this result is rather weak. First of all G could be identically equal to zero, in which case the function $\upsilon(\eta_1)$ would be identically zero also. Most people would not call this a bifurcation. The result does not tell how many r.e. are found since the method only looks for symmetric solutions. To overcome the first weakness only a little more computation needs to be carried out.

The full normalization of $\mathcal{U}$ was carried out by the method of Lie transforms using the second author's algebraic processor POLYPAK in almost all cases. The size of the problem grows rapidly with n since (1) the number of variables increases, (2) the number of critical masses increases, and (3) the order to which we must carry out the normalization increases. The first two cause linear growth in complexity whereas the third causes exponential growth in complexity. The full normalization is discussed in the next section.

If we are content to seek only solutions that are symmetric with respect to the x-axis we need only compute the first nonzero term in $G(\eta_1, 0, 0)$ to determine the general nature of the bifurcation. Thus the quest for symmetric solutions grows like a polynomial in n in the generic case.

Using the previous notation as found in (6) assume that

$$g(\eta_1, 0) = -\beta\eta_1^\rho + \cdots \tag{4.8}$$

where $\beta \neq 0$. Then the solution $\mu = \upsilon(\eta_1)$ is a solution of

$$\alpha\mu = g(\eta_1, \mu) = \alpha\mu - \beta\eta_1^\rho + \cdots = 0, \tag{4.9}$$

$$\mu = \upsilon(\eta_1) = \frac{\beta}{\alpha}\eta_1^\rho + \cdots. \tag{4.10}$$

Now we can decide how many symmetric relative equilibria bifurcate from the regular polygon relative equilibria as μ varies, since we can solve (4.6) for η_1 to find

$$\eta_1 = \sqrt[\rho]{\alpha\mu/\beta + \cdots}. \tag{4.11}$$

Here we use the standard convention about the ρth roots. In particular, if ρ is even, there are two r.e. that bifurcate from the r.p.r.e. for $\mu > 0$ when $\alpha\beta > 0$ or for $\mu < 0$ when $\alpha\beta < 0$. If ρ is odd one r.e. bifurcates from the r.p.r.e. for $\mu < 0$ and one for $\mu > 0$.

In the special case when n is even and $k = n/2 + 1$ the kernel of A has dimension 1 and so the splitting lemma says there are coordinates such that

$$\mathcal{U} = \pm\eta_2^2 \pm \eta_3^2 \pm \cdots \pm \eta_h^2 + G(\eta_1, \mu). \tag{4.12}$$

In the previous case we used the symmetry $\mathcal{R}$ to reduce the problem to that of solving (4.6). In this case we need only look at $\partial G/\partial\eta_1(\eta_1, \mu) = 0$ and proceed exactly as above.

Generically we would expect $\rho = 1$ unless the problem had a further symmetry in which case we would expect $\rho = 2$. We explain this difference in the next section. Thus in the generic case we do not have to compute the function G to high order. Appendix C contains a table of α, β, and ρ for both the N-body and the Kirchhoff problems for $3 \leqq n \leqq 12$, $2 \leqq k \leqq n/2+1$. Note that several entries are missing from the table for the N-body problem since these correspond to negative mass. The N-body problem behaves in a generic manner with ρ being 1 or 2, but the Kirchhoff problem is somewhat unpredictable. Note in Appendix C, when $n = 11$, $k = 6$ that $\rho = 2$ for the N-body problem whereas $\rho = 4$ for the Kirchhoff problem. The Kirchhoff problem is even more degenerate when $n = 4$, $k = 3$. In this case $\rho = 1$ and with it $g(\eta_1, \mu) = 8\mu(\eta_1 + \eta_1^3 + \cdots)$. All the terms have a factor μ and therefore the r.e. exists for $\mu = 0$ with η_1 arbitrary. We will come back to this case in the next section.

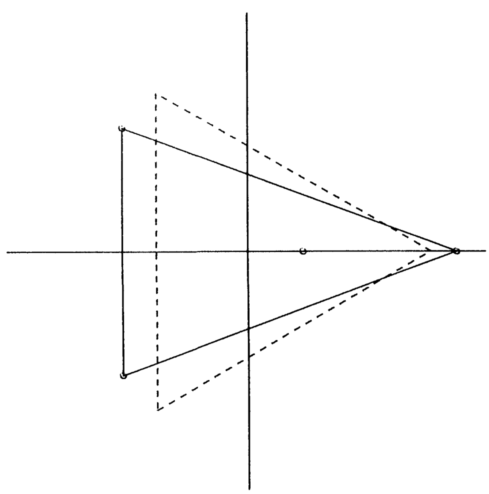

FIG. 1(a). $n = 3$, $k = 2$.

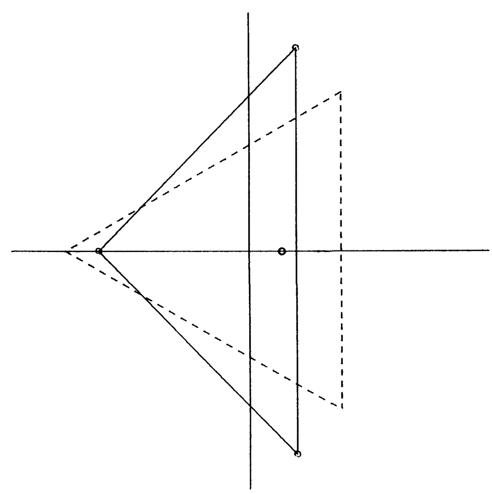

FIG. 1(b). $n = 3$, $k = 2$.

Figure 1 shows the r.e. which bifurcate from the equilateral triangle family. This is the case when $n = 3$, $k = 2$, and $\rho = 1$, so the two isosceles triangle r.e. exist on either side of the critical mass $m(3, 2) \simeq 0.77$ for the 4-body problem or $m(3, 2) = 1$ for the Kirchhoff problem. The acute triangle exists for $m < m(3, 2)$ and the obtuse for $m > m(3, 2)$. Figure 2 shows the r.e. which bifurcate from the square family when $n = 4$, $k = 3$, and $\rho = 2$. Only the kite r.e. shown in Figure 2(a) is symmetric with respect to the x-axis and is established by the above argument. It exists for $m > m(4, 3)$. Figure 3 shows all the r.e. that bifurcate from the duodecigon family when $n = 12$ and for various k and ρ. Only those shown in Figs. 3(a), 3(c), 3(e), 3(g), 3(i), 3(k) (every other one) are symmetric with respect to the x-axis or with respect to $\mathcal{R}$. Figure 3 shows the special case when n is even and $k = n/2+1$. These are the ones established by this argument. Note that all of the r.e. in Fig. 3 have an axis of symmetry even though in some cases it is difficult to see at first glance. We will discuss these figures more in the next section.

5. Symmetries and higher-order normalization. In the special case when n is even and $k = n/2+1$ the Hessian A does not have a two-dimensional kernel, and so the

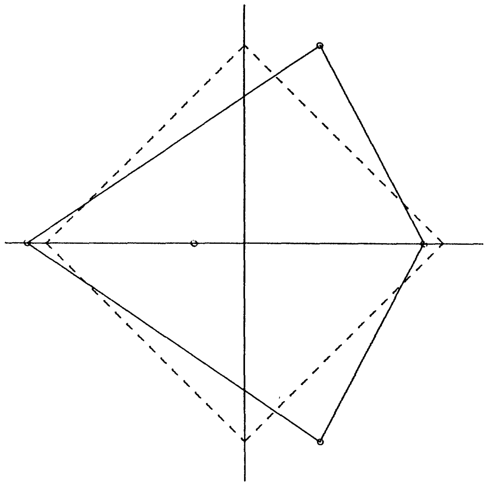

FIG. 2(a). $n=4$, $k=2$.

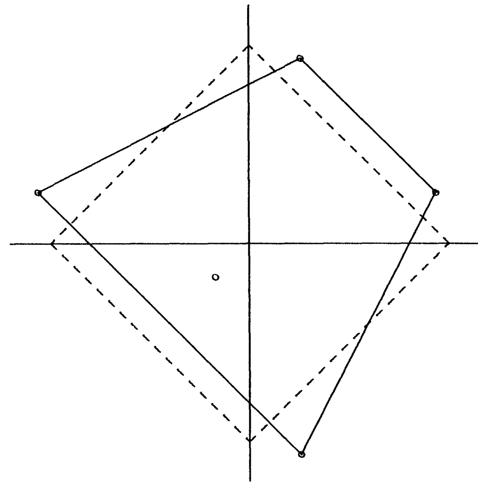

FIG. 2(b). $n=4$, $k=2$.

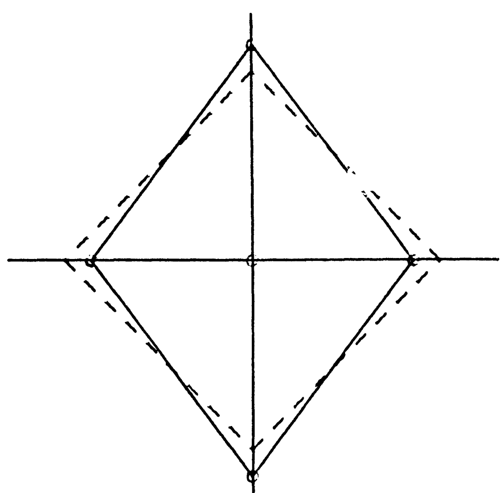

FIG. 2(c). $n=4$, $k=3$.

discussion of the previous section is complete for this case. Thus we will assume that $k \neq n/2+1$ throughout this section.

Let $\xi=(\xi_1, \cdots, \xi_h)^T=(x_2, \cdots, x_n, y_2, \cdots, y_n)^T$, where $h=2n-2$, are the Palmore coordinates discussed in § 2. As before fix n and k and let $\mu=m-m(n,k)$. Obviously $\mathcal{U}$ is invariant under the symmetries of the regular polygon with n sides; that is, there is a subgroup D_n of the orthogonal group $O(h, R)$ which is isomorphic to the dihedral group such that

$$\mathcal{U}(D\xi, \mu)=\mathcal{U}(\xi, \mu) \tag{5.1}$$

for all $D \in D_n$ and all small ξ and μ.

When $\mu=0$, the Hessian A of $\mathcal{U}$ at $\xi=0$ has a two-dimensional kernel and therefore there is an orthogonal matrix O, such that $O^TAO=\operatorname{diag}(0, 0, \lambda_3, \cdots, \lambda_h)$ where $\lambda_i \neq 0$ for $3 \leqq i \leqq h$. Let $O_2=\operatorname{diag}(1, 1, 1/\sqrt{|\lambda_3|}, \cdots, 1/\sqrt{|\lambda_h|})$ so $B=O^TAO=$

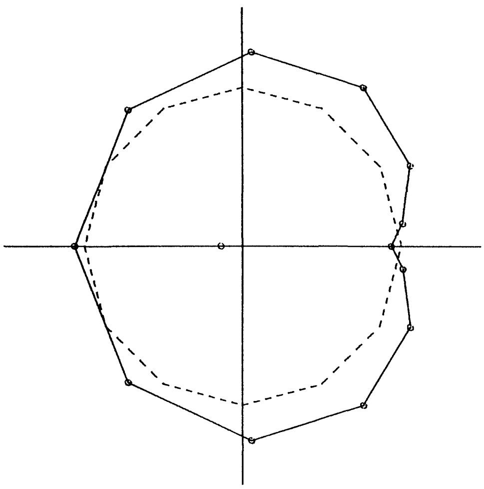

FIG. 3(a). $n = 12$, $k = 2$.

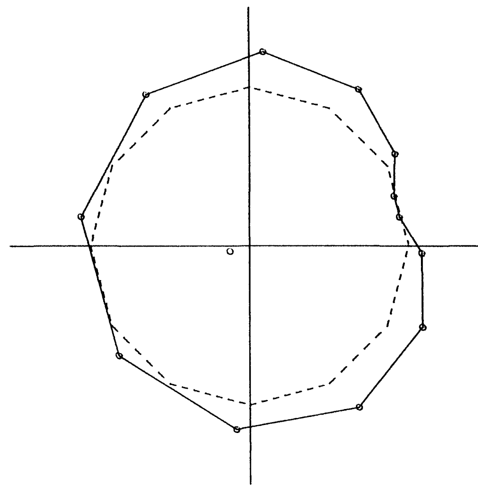

FIG. 3(b). $n = 12$, $k = 2$.

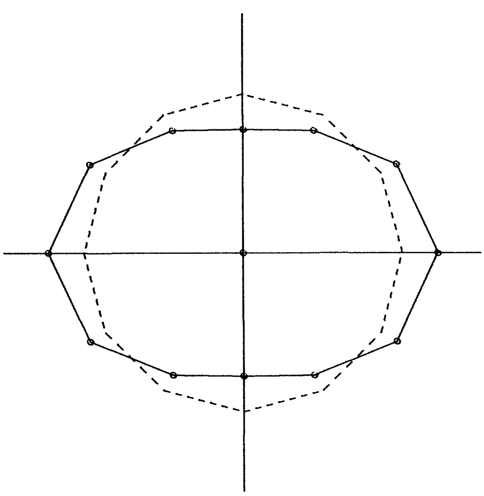

FIG. 3(c). $n = 12$, $k = 3$.

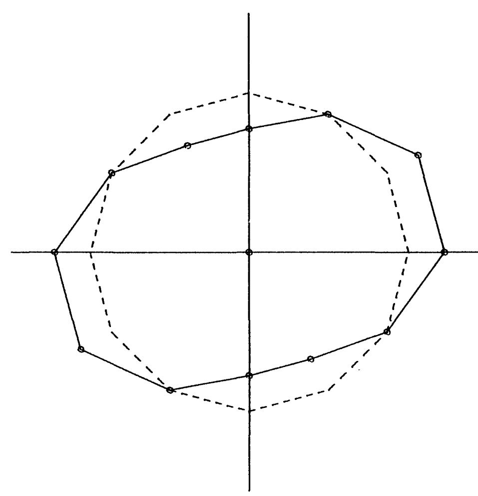

FIG. 3(d). $n = 12$, $k = 3$.

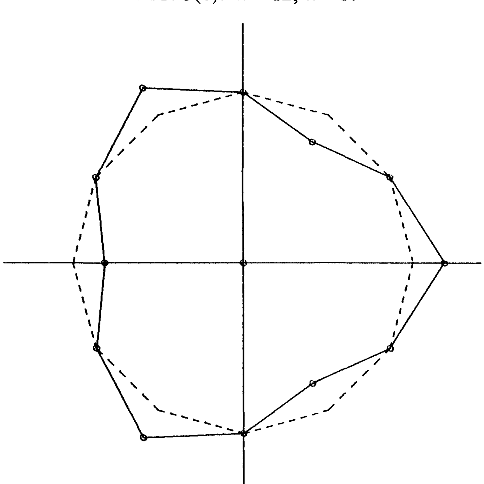

FIG. 3(e). $n = 12$, $k = 4$.

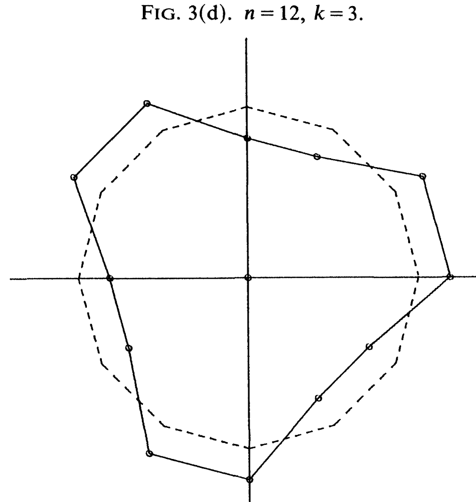

FIG. 3(f). $n = 12$, $k = 4$.

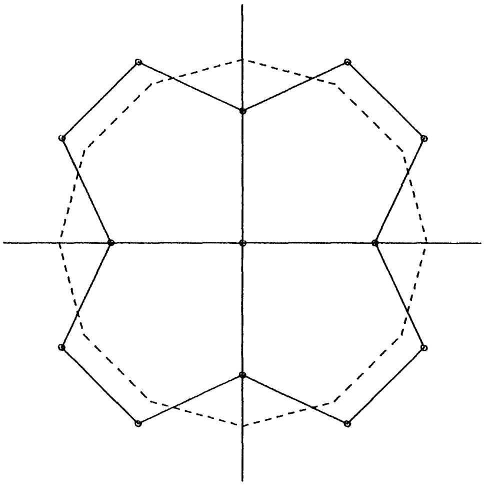

FIG. 3(g). $n=12$, $k=5$.

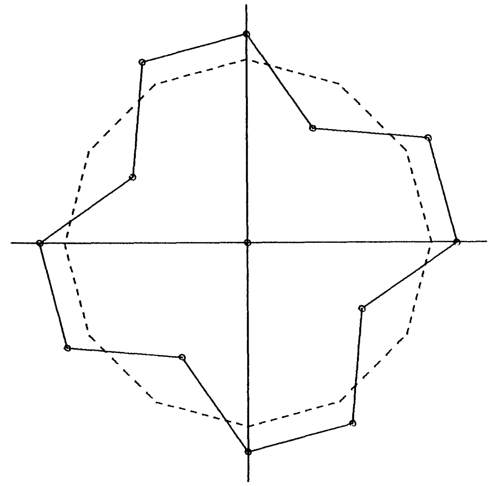

FIG. 3(h). $n=12$, $k=5$.

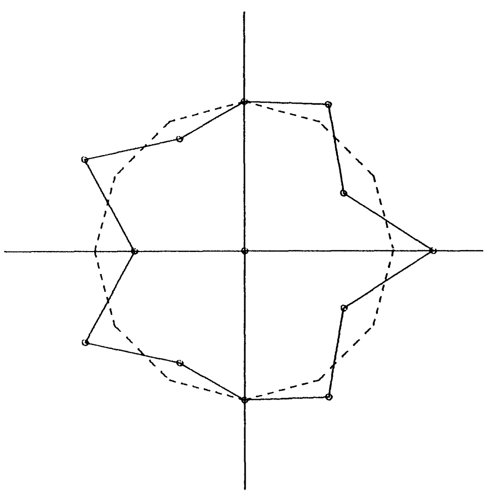

FIG. 3(i). $n=12$, $k=6$.

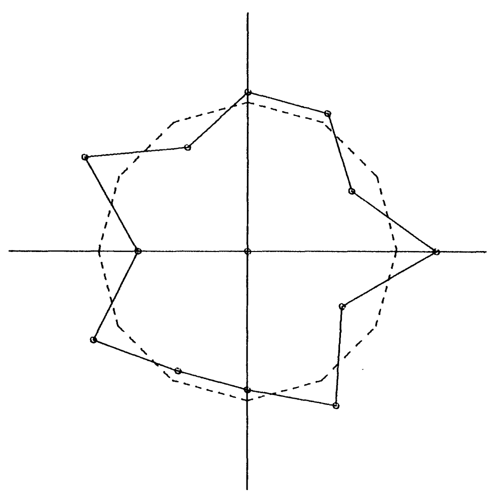

FIG. 3(j). $n=12$, $k=6$.

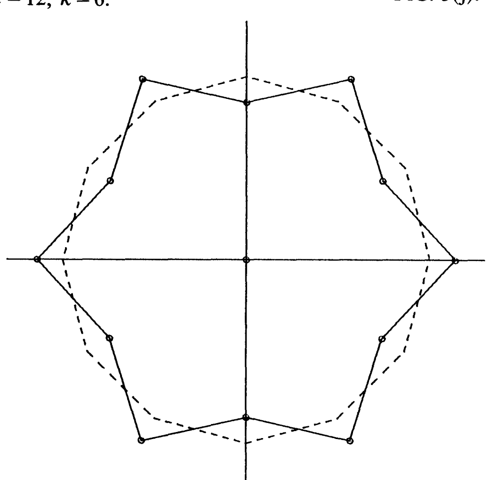

FIG. 3(k). $n=12$, $k=7$.

diag $(0, 0, \pm 1, \cdots, \pm 1)$ where $O = O_1 O_2$. If we change coordinates by $\xi = O\zeta$ then the Hessian of $\mathcal{U}$ in these coordinates is B or the quadratic part of $\mathcal{U}$ is as in (4.1). Usually, we use the same symbol for a function in different coordinates, but for the moment let $\mathcal{U}'(\zeta, \mu) = \mathcal{U}(O\zeta, \mu)$ so

$$\begin{aligned} \mathcal{U}(D\xi, \mu) &= \mathcal{U}(\xi, \mu), \\ \mathcal{U}(OO^{-1}DO\zeta, \mu) &= \mathcal{U}(O\zeta, \mu), \\ \mathcal{U}'(O^{-1}DO\zeta, \mu) &= \mathcal{U}'(\zeta, \mu), \end{aligned} \tag{5.2}$$

or

$$\mathcal{U}'(D'\zeta, \mu) = \mathcal{U}'(\zeta, \mu) \tag{5.3}$$

where $D' \in \mathcal{D}_n' = O^{-1}\mathcal{D}_n O$. From (5.3) D' leaves the Hessian of $\mathcal{U}'$ invariant and so $D'^T B D' = B$. This and the special form of O implies D' is of the form

$$D' = \begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix} \tag{5.4}$$

where E is a 2×2 orthogonal matrix and F is some nonsingular $(h-2) \times (h-2)$ matrix. The set of such E's form a subgroup $\mathscr{E}$ of $O(2, R)$ which is clearly isomorphic to the subgroup of $\mathcal{D}_n'$ obtained by letting F be the identity matrix in (5.4). Since $\mathscr{E}$ is isomorphic to a subgroup of a dihedral group and as we saw in the last section contains a reflection, it must be isomorphic to a dihedral group whose order divides $2n$. The order of this group depends on n and k, and the precise dependence will be given at the end of this section along with the discussion of the specific findings.

Let ε be a formal parameter and consider

$$\mathcal{U}_*(\zeta, \mu, \varepsilon) = \sum_{i=0}^{\infty} \left(\frac{\varepsilon^i}{i!}\right) \mathcal{U}_i^0(\zeta, \mu) \tag{5.5}$$

where $\mathcal{U}_*(\zeta, \mu, 1) = \mathcal{U}(\zeta, \mu)$ and $\mathcal{U}_i^0$ is a homogeneous polynomial in ζ of degree $i+2$. The method of Lie transform given by Deprit (1969) constructs a near identity change of variables

$$\zeta = \zeta(\eta, \mu, \varepsilon) = \eta + \cdots$$

where ζ is the general solution of the differential equation

$$\frac{d\zeta}{d\varepsilon} = W(\zeta, \mu, \varepsilon), \qquad \zeta|_{\varepsilon=0} = \eta. \tag{5.6}$$

If the function W has the formal expansion

$$W(\zeta, \mu, \varepsilon) = \sum_{l=0}^{\infty} \left(\frac{\varepsilon^l}{l!}\right) W_{l+1}(\zeta, \mu) \tag{5.7}$$

then in the new coordinates

$$\begin{aligned} \mathcal{U}^*(\eta, \mu, \varepsilon) &= \mathcal{U}_*(\zeta(\eta, \mu, \varepsilon), \mu, \varepsilon) \\ &= \sum_{j=0}^{\infty} \left(\frac{\varepsilon^j}{j!}\right) \mathcal{U}_0^j(\zeta, \mu). \end{aligned} \tag{5.8}$$

The functions $\mathcal{U}_*$ and $\mathcal{U}^*$ are related by the double index array $\{\mathcal{U}_i^j\}$, which agrees with the previous definitions when either i or j is zero and are related by the recursive relation

$$\mathcal{U}_i^j = \mathcal{U}_{i-1}^{j+1} + \sum_{k=0}^{i} \binom{i}{k} [\mathcal{U}_{i-k}^{j-1}, W_{k+1}], \tag{5.9}$$

where [,] is the Lie derivative operator on functions given by

$$[\mathcal{U}, W] = \frac{\partial \mathcal{U}}{\partial \zeta} W. \tag{5.10}$$

Let $\mathcal{P}_k$ be the space of homogeneous polynomials of degree $k+2$ in $\zeta_1, \cdots, \zeta_h$ with coefficients which are smooth in μ. Let $\mathcal{K}_k$ be the subspace of $\mathcal{P}_k$ of homogeneous polynomials in ζ_1 and ζ_2 only and $\mathcal{R}_k = \zeta_3 \mathcal{P}_{k-1} + \cdots + \zeta_h \mathcal{P}_{k-1}$ so that $\mathcal{P}_k = \mathcal{K}_k \oplus \mathcal{R}_k$. Since $\mathcal{U}_0^0 = \frac{1}{2}\zeta^T B \zeta$ the operator $L: W \to [\mathcal{U}_0^0, W]$ defines a self-map of $\mathcal{P}_k$ with kernel $\mathcal{K}_k$ and range $\mathcal{R}_k$. By a standard argument in normal form theory, we can find a formal series for W so that $\mathcal{U}^*$ is in normal form, i.e., $\mathcal{U}_0^k \in K_k$ for all $k \geqq 1$. That is, the higher-order terms in $\mathcal{U}^*$ depend only on η_1 and η_2. This argument is found in Meyer and Schmidt (1977), for example. This is the formal version of the splitting lemma.

Moreover, the normalizing generating function W satisfies $W_k \in \mathcal{R}_k$ so that the function W is zero on $\Xi = \{\xi: \xi_3 = \cdots = \xi_h = 0\}$. Thus Ξ is an invariant surface for (7) or the change of variables (6) fixes Ξ or $\Xi = \{\eta: \eta_3 = \cdots = \eta_h = 0\}$. This means that the new function $\mathcal{U}^*$ is invariant under the linear action defined by the matrices of the form (5.4) with $F = I$, the identity matrix.

We see that the normal form for $\mathcal{U}$ is the same as given by the splitting lemma in formula (4.1). Moreover, if the normalization is carried out as outlined above, the higher-order terms (i.e., G in 4.1) are invariant under the standard action of $\mathscr{C}$ on the plane.

Let $\mathscr{C}$ have order $2d$ where d divides n. Appendix C has a table giving d for various n and k. Consider the η_1, η_2 plane as the complex plane by setting $w = \eta_1 + i\eta_2$ and let $\phi = \exp(2\pi i/d)$ be a primitive dth root of unity. By the above, we are reduced to studying the critical points of

$$\Gamma(w, \bar{w}, \mu) = G(\eta_1, \eta_2, \mu) \tag{5.11}$$

where Γ is invariant under the action of $\mathscr{C}$ or

$$\Gamma(\phi w, \overline{\phi w}, \mu) = \Gamma(w, \bar{w}, \mu), \qquad \Gamma(w, \bar{w}, \mu) = \Gamma(\bar{w}, w, \mu). \tag{5.12}$$

The only terms in a Taylor expansion which satisfy the conditions in (5.12) are of the form

$$(w\bar{w})^i w^{dj} \quad \text{or} \quad (w\bar{w})^i \bar{w}^{dj} \tag{5.13}$$

where i and j are integers. Thus a typical expansion of Γ would look like this:

$$\Gamma(w, \bar{w}, \mu) = p_1(w\bar{w}) + p_2(w\bar{w})^2 + \cdots + (1/d) q_1(w^d + \bar{w}^d) + \cdots. \tag{5.14}$$

The p's and q's are real functions of μ. By the analysis of the Hessian given in § 3, $p_1(\mu) = a\mu + \cdots$ where a is a nonzero constant. Assume we are in the generic case so that $q_1(0) \neq 0$ and in addition $p_2(0) \neq 0$ when $d > 4$ and $p_2(0) \neq q_1(0)$ when $d = 4$.

Case 1. $d = 3$. Let $q_1(0) = b$, so we must solve

$$\begin{aligned} \frac{\partial \Gamma}{\partial w} &= a\mu\bar{w} + bw^2 + \cdots = 0, \\ a\mu(w\bar{w}) &+ bw^3 + \cdots = 0, \\ a\mu r^2 &+ br^3 \exp(i3\theta) + \cdots = 0 \end{aligned} \tag{5.15}$$

where $w = r\exp(i\theta)$. Thus, by the implicit function theorem Γ has critical points at $r = \mp a\mu/b + \cdots$ when $\exp(i3\theta) = \pm 1$. That is, three critical points move linearly away from the origin (or the r.p.r.e.) as μ varies from zero. This case occurs when $n = 3$, $k = 2$ (Fig. 1(a) shows a solution where $\exp(i3\theta) = +1$ and Fig. 1(b) where $\exp(i3\theta) = -1$.); when $n = 9$, $k = 4$; $n = 12$, $k = 5$ (Fig. 3(g) shows a solution where $\exp(i3\theta) = +1$ and Fig. 3(h) where $\exp(i3\theta) = -1$) for both problems, and when $n = 6$, $k = 3$ for the Kirchhoff problem. See Appendix C. In the notation of the previous section $\alpha = a$, $\beta = b$, and $\rho = 1$.

Case 2. $d \geqq 5$. Let $p_2(0) = b \neq 0$ and $q_1(0) = c \neq 0$, so we must solve

$$\begin{gathered}\frac{\partial \Gamma}{\partial w} = a\mu\bar{w} + b(w\bar{w})^2 + \cdots + cw^{d-1} + \cdots = 0,\\ a\mu r^2 + br^4 + \cdots + cr^d \exp(id\theta) + \cdots = 0.\end{gathered} \tag{5.16}$$

By the implicit function theorem Γ has critical points at

$$r = \sqrt{-a\mu/b + \cdots}, \qquad \exp(id\theta) = \pm 1. \tag{5.17}$$

That is, Γ has $2d$ nonzero critical points for $\mu > 0$ and none for $\mu < 0$ when $ab < 0$ and vice versa when $ab > 0$. These solutions fall into two families of d each depending on the sign of $\exp(id\theta)$. The families move away from the origin like the square root of μ. For most n and k, we have $d \geqq 5$ (see Appendix C). For $n = 12$, Figs. 3(a), 3(i) show the solutions where $\exp(i12\theta) = +1$ and Figs. 3(b), 3(j) show the solutions where $\exp(i12\theta) = -1$. In the notation of the previous section $\alpha = a$, $\beta = b$, and $\rho = 2$.

Case 3. $d = 4$. Using the above notation, we must solve

$$a\mu r^2 + (b + c\exp(i4\theta))r^4 + \cdots = 0 \tag{5.18}$$

and so there are solutions of the form

$$r = \sqrt{-a\mu/(b \pm c) + \cdots}, \qquad \exp(i4\theta) = \pm 1. \tag{5.19}$$

If $b \pm c$ are of one sign then there are eight solutions for μ on one side of zero as in Case 2. This happens when $n = 12$, $k = 4$. Figure 3(e) shows a solution when $\exp(i4\theta) = +1$ and Fig. 3(f) shows a solution where $\exp(i4\theta) = -1$. If $b \pm c$ have two signs then there are four solutions when μ is negative and when μ is positive as in Case 1. This happens when $n = 4$, $k = 2$. Figure 2(a) shows a solution when $\exp(i4\theta) = +1$ and Fig. 2(b) shows a solution when $\exp(i4\theta) = -1$.

To understand the relationship between n, k, and the order d of the rotational subgroup which acts on the two-dimensional subspace, we proceed as we did in the previous section when we discussed the reflection symmetry. $\mathcal{U}$ is invariant also under a rotation $\mathcal{O}$ which leaves the regular polygon relative equilibrium fixed. In the original coordinates the rotation is

$$\mathcal{O}: q_j \to \omega q_{j-1}, \quad 0 \leqq j < n, \qquad \mathcal{O}: q_n \to q_n. \tag{5.20}$$

This rotation $\mathcal{O}$ with the reflection $\mathcal{R}$ generates the symmetry group of $\mathcal{U}$ which also fixes the r.p.r.e. Ω. At one of the critical masses a perturbation in the direction of the kernel of the Hessian is of the form

$$\begin{gathered} q_j = \omega^j + \omega^{jk} z_k + \omega^{jl} z_l, \qquad k + l = n + 2,\\ q_j = \omega^j + \omega^{jk} z, \qquad k = l = n/2 + 1.\end{gathered} \tag{5.21}$$

In the first case the z_k and z_l are not independent but are linearly related (essentially conjugates), so one can be used as a coordinate of the perturbation. In the second case the z is arbitrary. Thus we can use z_k or z as a coordinate in the kernel of the Hessian. The action of $\mathcal{O}$ on this subspace is

$$\begin{aligned}&\mathcal{O}: \omega^j+\omega^{jk}z_k+\omega^{jl}z_l \to \omega^j+\omega^{jk}(\omega^{1-k})z_k+\omega^{jl}(\omega^{l-1})z_l, \qquad k\neq l,\\ &\mathcal{O}: \omega^j+\omega^{jk}z \to \omega^j+\omega^{jk}(\omega^{1-k})z, \qquad k=l=n/2+1.\end{aligned} \tag{5.22}$$

Thus in coordinates $\mathcal{O}: z_k \to (\omega^{1-k})z_k$ or $\mathcal{O}: z\to(\omega^{1-k})z$, so $\mathcal{O}$ is a rotation on this subspace also; but it does not necessarily generate the full symmetry group. The order of the rotation group generated by $\mathcal{O}$ on this subspace is d where $(k-1)d\equiv 0 \bmod n$. Appendix C lists n, k, and d for all cases $3\leqq n\leqq 12$.

Consider Figs. 3(e) and 3(f) for example where $n=12$, $k=4$, and $d=4$. By rotating these figures by $l2\pi/12$ for $l=0, 1, \cdots, 11$ we obtain $d=4$ distinct r.e. which have symmetry given by the dihedral group D_3, because $k-1=3$. Contrast that with Figs. 3(a) and 3(b) where $n=12$, $k=2$, $d=12$. When we rotate these figures by $l2\pi/12$, $l=0,\cdots, 11$ we obtain $d=12$ distinct r.e., which have the symmetries of the dihedral group D_1 (generated by a single reflection), since $k-1=1$. The other figures follow the same pattern.

Finally we will consider the degeneracy in the Kirchhoff problem when $n=4$, $k=3$. In this case the symmetry with respect to both axes is preserved. Since the moment of inertia has to remain constant, r.e. can only be formed by a rhombus with the fifth vortex at the center. The coordinates of the five vortices are

$$\begin{aligned}&q_0=-q_2=1+x_0,\\ &q_1=q_3=i\sqrt{1-2x_0-x_0^2},\\ &q_4=0.\end{aligned}$$

Let m be the value of the vorticity at the origin, then the potential function (2.8) turns out to be

$$-\tfrac{7}{8}\log 2-(1+2m)\{\log(1+x_0)+\tfrac{1}{2}\log(1-2x_0-x_0^2)\}.$$

For $m\neq-\frac{1}{2}$ the potential function has extrema at $x_0=0$ and $x_0=-\frac{1}{2}$. For $m=-\frac{1}{2}$ the potential function is independent of x_0 and therefore any rhombus can serve as a r.e. for the Kirchhoff problem. See Fig. 2(c).

Appendix A. Entries in the Hessian.

$$B=\begin{pmatrix} b_2 & & & \\ & b_3 & & \\ & & \ddots & \\ & & & b_n\end{pmatrix}, \qquad C=C^T=\begin{pmatrix} & & & c_n\\ & & \iddots & \\ & c_3 & & \\ c_2 & & & \end{pmatrix},$$

$$b_k=\frac{n}{2}(-R_k-\gamma m), \qquad k=2,3,\cdots,n-1,$$

$$b_n=\frac{n}{2}\frac{m+n}{m}(-\gamma m-\delta n-S),$$

$$c_k=\frac{n}{2}(T_k-\gamma m), \qquad k=3,4,\cdots,n-1,$$

$$c_k=-\frac{n}{2}(m+n)\gamma, \qquad k=2, n$$

where $\gamma = \delta + 2$ and

$$S = \frac{1}{2^\delta} \sum_{r=1}^{n-1} \frac{1}{\sin^\delta (\pi r/n)},$$

$$R_k = \frac{\delta}{2^{\delta+1}} \sum_{r=1}^{n-1} \frac{\sin^2 (\pi r k/n)}{\sin^\gamma (\pi r/n)} + S,$$

$$T_k = \frac{\gamma}{2^{\delta+1}} \sum_{r=1}^{n-1} \frac{\sin (\pi r k/n) \sin (\pi r(k-2)/n)}{\sin^\gamma (\pi r/n)},$$

$$m(n, 2) = \frac{R_k(\delta n + S)}{\gamma(2n - R_2 - S)},$$

$$m(n, k) = \frac{T_k^2 - R_k R_l}{\gamma(R_k + R_l + 2T_k)}, \qquad k + l = n + 2, \quad k \neq l,$$

$$m(n, k) = \frac{T_k - R_k}{2\gamma}, \qquad 2k = n + 2.$$

In the Kirchhoff problem, $\delta = 0$, the formulas for m simplify to

$$m(n, 2) = \tfrac{1}{4}(n-1)^2,$$

$$m(n, k) = \tfrac{1}{4}\{(k-2)(n-k) - n + 1\}, \qquad k = 3, 4, \cdots, (n+2)/2.$$

Appendix B. Critical masses and subdeterminants.

		Kirchhoff		$n+1$ body problem	
n	k	$m(n, k)$	$d(n, k)$	$m(n, k)$	$d(n, k)$
3	2	1.000E+00	—	7.705E−01	—
4	2	2.250E+00	—	2.380E+00	—
4	3	−5.000E−01	2.000E+00	−2.500E−01	1.500E+00
5	2	4.000E+00	—	6.478E+00	—
5	3	−5.000E−01	1.200E+01	−2.442E−01	1.144E+01
6	2	6.250E+00	—	2.091E+01	—
6	3	−5.000E−01	1.600E+01	−2.201E−01	1.577E+01
6	4	−2.500E−01	1.000E+00	5.983E−03	−3.590E−02
7	2	9.000E+00	—	−6.433E+02	—
7	3	−5.000E−01	2.000E+01	−1.814E−01	1.800E+01
7	4	0.0	0.0	3.242E−01	−4.342E+01
8	2	1.225E+01	—	−3.793E+01	—
8	3	−5.000E−01	2.400E+01	−1.306E−01	1.686E+01
8	4	2.500E−01	−1.500E+01	6.980E−01	−1.301E+02
8	5	5.000E−01	−2.000E+00	9.963E−01	−5.978E+00
9	2	1.600E+01	—	−2.544E+01	—
9	3	−5.000E−01	2.800E+01	−6.937E−02	1.116E+01
9	4	5.000E−01	−3.600E+01	1.119E+00	−2.725E+02
9	5	1.000E+00	−8.000E+01	1.774E+00	−5.133E+02
10	2	2.025E+01	—	−2.172E+01	—
10	3	−5.000E−01	3.200E+01	1.064E−03	−2.068E−01
10	4	7.500E−01	−6.300E+01	1.581E+00	−4.819E+02
10	5	1.500E+00	−1.440E+02	2.641E+00	−1.002E+03
10	6	1.750E+00	−7.000E+00	3.012E+00	−1.807E+01
11	2	2.500E+01	—	−2.027E+01	—

Appendix B. Critical masses and subdeterminants (*cont.*)

11	3	−5.000E−01	3.600E+01	7.969E−02	−1.830E+01
11	4	1.000E+00	−9.600E+01	2.080E+00	−7.687E+02
11	5	2.000E+00	−2.240E+02	3.588E+00	−1.708E+03
11	6	2.500E+00	−3.000E+02	4.391E+00	−2.340E+03
12	2	3.025E+01	—	−1.974E+01	—
12	3	−5.000E−01	4.000E+01	1.657E−01	−4.411E+01
12	4	1.250E+00	−1.350E+02	2.611E+00	−1.143E+03
12	5	2.500E+00	−3.200E+02	4.605E+00	−2.665E+03
12	6	3.250E+00	−4.550E+02	5.894E+00	−3.949E+03
12	7	3.500E+00	−1.400E+01	6.338E+00	−3.803E+01

Appendix C. Coefficients.

			Kirchhoff			$n+1$ body problem		
n	k	d	ρ	α	β	ρ	α	β
3	2	3	1	−1.200E+00	1.717E+01	1	−1.921E+00	1.562E+01
4	2	4	2	−7.843E−01	−4.919E+01	2	−9.756E−01	−5.327E+01
4	3	3	1	8.000E+00	0	—	—	—
5	2	5	2	−5.769E−01	5.858E+00	2	−3.773E−01	1.132E+01
5	3	5	2	1.000E+01	−7.500E+00	—	—	—
6	2	6	2	−4.541E−01	5.456E+00	2	−7.906E−02	8.977E+00
6	3	3	1	1.200E+01	−1.697E+01	—	—	—
6	4	4	2	1.200E+01	−4.500E+01	2	1.800E+01	−7.850E+01
7	2	7	2	−3.733E−01	5.312E+00	—	—	—
7	3	7	2	1.400E+01	−2.100E+02	—	—	—
7	4	7	?	1.400E+01	??	2	2.098E+01	−1.169E+02
8	2	8	2	−3.165E−01	5.298E+00	—	—	—
8	3	4	2	1.600E+01	−3.840E+02	—	—	—
8	4	8	2	1.600E+01	9.000E+01	2	2.393E+01	−1.930E+02
8	5	4	2	1.600E+01	−2.560E+02	2	2.400E+01	−5.497E+02
9	2	9	2	−2.744E−01	5.360E+00	—	—	—
9	3	9	2	1.800E+01	−2.100E+02	—	—	—
9	4	3	1	1.800E+01	−5.728E+01	1	2.688E+01	−6.443E+01
9	5	9	2	1.800E+01	−2.550E+02	2	2.699E+01	−6.070E+02
10	2	10	2	−2.420E−01	5.469E+00	—	—	—
10	3	5	2	2.000E+01	−2.400E+02	2	2.930E+01	−2.550E+02
10	4	10	2	2.000E+01	−1.102E+03	2	2.982E+01	−1.325E+03
10	5	10	2	2.000E+01	−4.800E+02	2	2.997E+01	−1.685E+03
10	6	4	2	2.000E+01	−8.750E+02	2	3.000E+01	−2.253E+03
11	2	11	2	−2.164E−01	5.609E+00	—	—	—
11	3	11	2	2.200E+01	−2.772E+02	2	3.217E+01	−3.099E+02
11	4	11	2	2.200E+01	−9.900E+02	2	3.276E+01	−1.479E+03
11	5	11	4	2.200E+01	3.831E+04	2	3.294E+01	−2.005E+03
11	6	11	2	2.200E+01	−7.425E+02	2	3.299E+01	−2.312E+03
12	2	12	2	−1.956E−01	5.772E+00	—	—	—
12	3	6	2	2.400E+01	−3.200E+02	2	3.504E+01	−3.739E+02
12	4	4	2	2.400E+01	−1.836E+03	2	3.570E+01	−2.628E+03
12	5	3	1	2.400E+01	−1.357E+02	1	3.591E+01	−1.759E+02
12	6	12	2	2.400E+01	−1.925E+03	2	3.598E+01	−6.147E+03
12	7	4	2	2.400E+01	−2.304E+03	2	3.600E+01	−6.947E+03

Appendix D. The potential of the hexagon configuration. Consider the one-parameter perturbation of the hexagon configuration (with no particle at the centroid) in the n-body problem given by

$$q_0=(1+\varepsilon)(2,0), \qquad q_1=\sqrt{(1-2\varepsilon-\varepsilon^2)}(1,\sqrt{3}),$$
$$q_2=(1+\varepsilon)(-1,\sqrt{3}), \qquad q_3=\sqrt{(1-2\varepsilon-\varepsilon^2)}(-2,0),$$
$$q_4=(1+\varepsilon)(-1,-\sqrt{3}), \qquad q_5=\sqrt{(1-2\varepsilon-\varepsilon^2)}(1,-\sqrt{3}).$$

This perturbation has been chosen to keep the moment of inertia, I, fixed. From the symmetry,

$$\begin{aligned} U&=\frac{6}{\|q_0-q_1\|}+\frac{3}{\|q_0-q_2\|}+\frac{3}{\|q_1-q_3\|}+\frac{3}{\|q_0-q_3\|}\\ &=3\{1-\varepsilon^2+\cdots\}+\frac{\sqrt{3}}{2}\{1-\varepsilon+\varepsilon^2+\cdots\}+\frac{\sqrt{3}}{2}\{1+\varepsilon+2\varepsilon^2+\cdots\}+\frac{3}{4}\left\{1+\frac{1}{2}\varepsilon^2+\cdots\right\}\\ &=\left(\frac{15}{4}+\sqrt{3}\right)-\frac{3}{8}(7-4\sqrt{3})\varepsilon^2+\cdots\\ &\simeq 5.48-0.0269\varepsilon^2+\cdots. \end{aligned}$$

Thus U initially decreases along this family and so the hexagon configuration is not a minimum of the (self-) potential.

REFERENCES

A. DEPRIT (1969), *Canonical transformation depending on a small parameter*, Celestial Mech., 72, pp. 173–179.

O. DZIOBEK (1900), *Über einen merkwürdigen Fall des Vielkörperproblem*, Astronom. Nachr., 152, pp. 32–46.

L. EULER (1767), *De motu restilineo trium corporum se mutus attrahentium*, Novi Comm. Acad. Sci. Imp. Petrop., 11, pp. 144–151.

M. GOLUBITSKY AND D. G. SCHAEFFER (1985), *Singularities and Groups in Bifurcation Theory*, Springer-Verlag, New York.

E. HOPF (1942), *Abzweigung einer periodschen Lösung von einer stationären Lösung eines Differential System*, Ber. Verh. Sachs. Acad. Wiss. Leipzig Math.-Nat. Kl., 95(1), pp. 3–22.

R. HOPPE (1879), *Erweiterung der bekannten Speciallösung des Dreikörper Problems*, Archiv Math. und Phys., 64, pp. 218–223.

G. KIRCHHOFF (1897), *Vorlesungen über Mechanik*, Druck und Verlag von B. G. Teubner, Leipzig.

J. L. LAGRANGE (1772), *Essai sur le problem des trois corps*, Oeuvers, 6, pp. 272–292.

R. LEHMANN-FILHES (1891), *Über zwei Fälle des Vielkörpersproblem*, Astronom. Nachr., 127, pp. 137–144.

K. R. MEYER AND D. S. SCHMIDT (1977), *Entrainment domains*, Funkcial. Ekvac., 20, pp. 171–192.

——— (1988), *Bifurcations of relative equilibria in the* 4 *and* 5 *body problems*, to appear in Ergodic Theory, Dynamical Systems.

R. MOECKEL (1985), *Relative equilibria of the four-body problem*, Ergodic Theory Dynamical Systems, 5, pp. 417–435.

F. R. MOULTON (1910), *The straight line solutions of the problem of n-bodies*, Bull. Amer. Math. Soc., 13, pp. 324–335.

J. I. PALMORE (1973), *Relative equilibria of the n-body problem*, Ph.D. thesis, University of California, Berkeley, CA.

——— (1975), *Classifying relative equilibria* II, Bull. Amer. Math. Soc., 81, pp. 489–491.

——— (1976), *Measure of degenerate relative equilibria* I, Ann. of Math., 140, pp. 421–429.

——— (1982), *Relative equilibria of vortices in two dimensions*, Proc. Nat. Acad. Sci. USA, 79, pp. 716–718.

T. POSTON AND I. STEWART (1978), *Catastrophe Theory and its Applications*, Pitman, Boston.

C. SIMO (1977), *Relative equilibrium solutions in the four body problem*, Celestial Mech., 18, pp. 165–184.

S. SMALE (1970), *Topology and mechanics.* II, *The planar n-body problem*, Invent. Math., 11, pp. 45–64.

D. SAARI (1980), *On the pole and properties of n-body central configurations*, Celestial Mech., 21, pp. 9–20.