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ON THE METHOD OF AVERAGING, INTEGRAL MANIFOLDS AND SYSTEMS WITH SYMMETRY*

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Abstract. A unified treatment and generalizations of some of the more important results associated with the method of averaging are given. The generalizations make the results applicable to a larger class of problems. Furthermore, the method of proof is simpler than what is traditional in this field. Examples demonstrating the use of basic results in applications are given.

Key words. method of averaging, integral manifolds, symmetry

1. Introduction. We present here a unified treatment of some of the more important results associated with the method of averaging and in addition generalize them to the case of systems with symmetry. Our results, furthermore, are not restricted to systems that are almost periodic in time and therefore are applicable to a larger class of systems as they occur in applications.

Our approach is mathematically simpler than what is traditional and therefore perhaps more accessible to individuals in the applied fields. This simplicity is attained, however, at a price. We are here content to obtain estimates on size and location of certain invariant sets of the dynamical systems while the traditional results, in many cases, give a description of the motion in the invariant set.

The results given here generalize those by Sethna in [1] to the case of invariant manifolds.

Consider the system

$$\dot{u} = Au + \varepsilon h(t, u, \varepsilon) \tag{1.1}$$

where $u \in R^n$, A is $n \times n$ constant matrix, ε is a real parameter and the function $h: R \times R^n \times [0, \varepsilon_0] \to R^n$ is smooth, the properties of which will be discussed below. The parameter ε is often introduced by "scaling" the dependent variable in some appropriate manner. The matrix A will be assumed to have eigenvalues on the pure imaginary axis and all eigenvalues are assumed to have simple elementary divisors. Thus the system is "critical" in the sense of Hale [2]. The restriction that all the eigenvalues lie on the pure imaginary axis is in fact not a restriction if we regard u as on the center manifold [3]. The method of averaging can be applied to classes of problems distinct from those represented by (1.1) where the parameter ε does not arise from "scaling" and thus the results then are not restricted to local neighborhoods of the state space [4]. Our results will apply also to such problems.

We first reduce (1.1) to "standard form." Let $u = e^{At}x$ where $x \in R^n$. Then by hypothesis e^{-At} exists for all t and x satisfies

$$\dot{x} = \varepsilon f(t, x, \varepsilon) \quad \text{where } f(t, x, \varepsilon) = e^{-At}h(t, e^{At}x, \varepsilon). \tag{1.2}$$

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The function f is assumed to be a smooth vector valued function of its variables and in the standard results f is assumed to be almost periodic in t. In this introduction, for simplicity, we will assume f to be merely a Fourier polynomial in t.

For system (1.2), the standard results in the simplest case are based on the following change of variables [5].

Let

$$x = \xi + \varepsilon\phi(t, \xi) \tag{1.3}$$

where $\xi \in R^n$ and

$$\phi(t, \xi) \equiv \int^t [f(s, \xi, 0) - f_0(\xi)]\, ds \tag{1.4}$$

where

$$f_0(\xi) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t, \xi, 0)\, dt. \tag{1.5}$$

Substituting (1.3) into (1.2), we have

$$\left(I + \varepsilon\frac{\partial\phi}{\partial\xi}\right)\dot{\xi} = \varepsilon f_0(\xi) + \varepsilon[f(t, \xi + \varepsilon\phi, \varepsilon) - f(t, \xi, 0)] \tag{1.6}$$

where we have eliminated $\partial\phi/\partial t$ by using (1.4) and where I is the $n \times n$ identity matrix.

By fixing a large radius R, the matrix $\partial\phi/\partial\xi$ is bounded for all $|\xi| \leqq R$ and so $(I + \varepsilon\, \partial\phi/\partial\xi)$ has an inverse for $|\xi| \leqq R$ and $|\varepsilon| \leqq \varepsilon_0$ for some $\varepsilon_0 > 0$. Moreover, this inverse is of the form $I + \varepsilon\Psi(t, \xi, \varepsilon)$ where Ψ is uniformly bounded for $|\xi| \leqq R$, $|\varepsilon| \leqq \varepsilon_0$. We wish to investigate the solutions of these equations near the zeros of f_0. Let the set of zeros of f_0 be denoted by M and assume that M is closed and bounded. Thus the radius R should be taken so large so that M lies inside the sphere of radius R and this choice of R fixes ε_0.

Equations (1.6) can then be written as

$$\dot{\xi} = \varepsilon f_0(\xi) + \varepsilon^2\tilde{g}(t, \xi, \varepsilon) \tag{1.7}$$

where

$$\tilde{g} = \Psi(t, \xi, \varepsilon)f_0(\xi) + \varepsilon^{-1}[f(t, \xi + \varepsilon\phi, \varepsilon) - f(t, \xi, \varepsilon)] + O(\varepsilon). \tag{1.8}$$

From (1.8) we see that $\tilde{g}$ is smooth and uniformly bounded for all t, $|\xi| \leqq R$ and $|\varepsilon| \leqq \varepsilon_0$.

The "averaged system" corresponding to (1.2) is defined to be the autonomous system

$$\dot{\xi} = \varepsilon f_0(\xi) \tag{1.9}$$

and the standard results for the method of averaging give information about the solutions of (1.2) based on the properties of solutions of (1.9).

For results valid for all t, i.e. for what are called "steady state" solutions in applications, the simplest result is as follows. It is assumed that (1.9) has a constant solution ξ^0 and that the matrix $\partial f_0(\xi^0)/\partial\xi$ has eigenvalues with nonzero real parts; then it is shown that there exists an ε^* such that for $0 < \varepsilon < \varepsilon^*$, there exists a solution $x(t, \varepsilon)$ of (1.2) which is almost periodic in t (if f is almost periodic in t) such that $x(t, \varepsilon) \to \xi^0$ as $\varepsilon \to 0$ and $x(t, \varepsilon)$ has the same stability properties as ξ^0. The results given in this work are generalizations of those in [1]. The difference in the nature of the standard result and those obtained here can now be explained in this simple case.

Results in [1] differ from the above standard result in that in [1] f is assumed, with some restrictions, to be merely bounded by $t > t_0$ and the result given proves that there exists an ε^*, such that for ε, $0 < \varepsilon < \varepsilon^*$ there is a positively invariant set for (1.2) in a small neighborhood of ξ^0 if the eigenvalues of $\partial f_0(\xi^0)/\partial\xi$ have negative real parts and that some solution of (1.2) leaves a small neighborhood of ξ^0 if $\partial f_0(\xi^0)/\partial\xi$ has at least one eigenvalue with positive real part. In each case the neighborhood shrinks to ξ^0 as $\varepsilon \to 0$.

The crucial and limiting condition in all these results is the condition on the eigenvalues of the matrix $\partial f_0(\xi^0)/\partial\xi$. In many instances the constant solutions of (1.9) are not isolated but form invariant sets, usually surfaces, that are subsets of R^n.

Consider for example the van der Pol equation

$$\begin{aligned} \dot{u}_1 &= \omega u_2, \\ \dot{u}_2 &= -\omega u_1 + \varepsilon(1 - u_1^2)u_2. \end{aligned} \tag{1.10}$$

The principal matrix solution for $\varepsilon = 0$ is

$$e^{At} = \begin{bmatrix} \sin \omega t & \cos \omega t \\ \cos \omega t & -\sin \omega t \end{bmatrix} \tag{1.11}$$

and the transformation $u = (u_1, u_2) = e^{At}x$ where $x = (x_1, x_2)$ leads to

$$\begin{aligned} \dot{x}_1 &= \varepsilon(1 - u_1^2)u_2 \cos \omega t, \\ \dot{x}_2 &= \varepsilon(1 - u_1^2)u_2 \sin \omega t. \end{aligned} \tag{1.12}$$

Now if we introduce the transformation $x \to \xi$ as in (1.3), system (1.12) reduces to

$$\begin{aligned} \dot{\xi}_1 &= \frac{\varepsilon}{2}\left(1 - \frac{\xi_1^2 + \xi_2^2}{4}\right)\xi_1 + \varepsilon^2 g_1(t, \xi_1, \xi_2, \varepsilon), \\ \dot{\xi}_2 &= \frac{\varepsilon}{2}\left(1 - \frac{\xi_1^2 + \xi_2^2}{4}\right)\xi_2 + \varepsilon^2 g_2(t, \xi_1, \xi_2, \varepsilon) \end{aligned} \tag{1.13}$$

with $\tilde{g}$ as in (1.8). The averaged system is the same as (1.13) with $\tilde{g}_1 = \tilde{g}_2 = 0$. The constant solutions of the averaged system are solutions of

$$f_0(\xi) = \begin{bmatrix} \xi_1 - \left(\dfrac{\xi_1^2 + \xi_2^2}{4}\right)\xi_1 \\ \xi_2 - \left(\dfrac{\xi_1^2 + \xi_2^2}{4}\right)\xi_2 \end{bmatrix} = 0. \tag{1.14}$$

The solution $\xi^0 = (0, 0)$ is isolated and the corresponding variational equation has positive real eigenvalues. On the other hand (1.14) has the family of solutions $(\xi_1^{0^2} + \xi_2^{0^2}) = 4$, each of which has a zero eigenvalue. This can be seen as follows. Introduce the parameter θ with $\xi_1^0 = 2 \cos \theta$, $\xi_2^0 = 2 \sin \theta$. Then

$$\frac{\partial f_0(\xi^0)}{\partial \xi} \frac{d\xi^0}{d\theta} = \frac{1}{2}\begin{bmatrix} -\xi_1^{0^2} & -\xi_1^0\xi_2^0 \\ \xi_1^0\xi_2^0 & \xi_2^{0^2} \end{bmatrix}\begin{pmatrix} -2 \sin \theta \\ 2 \cos \theta \end{pmatrix} = 0.$$

Thus any solution on the circle $\xi_1^{0^2} + \xi_2^{0^2} = 4$ has a zero eigenvalue and the corresponding eigenvector $d\xi^0/d\theta$ is tangent to the circle.

Such behavior can occur on tori of any dimension less than n and the standard results of the method of averaging treat such problems by introducing suitable polar coordinates in the formulation of the problem, so that the original system in standard form is periodic in a vector variable θ. Such results are discussed at length in [6].

We wish to demonstrate below that analogous behavior also occurs in systems with symmetry. Suppose the system (1.1) with $u \in E^n$, an n-dimensional Euclidian space, has k parameter symmetry, i.e. there exists an $n \times n$ nonsingular matrix $S(\theta)$, $\theta \in R^k$ such that $S(0) = I$, $\partial S(0)/\partial\theta \neq 0$ and is C^1 for all $\theta \in R^k$ and

$$S(\theta)A = AS(\theta), \qquad S(\theta)h(t, u, \varepsilon) = h(t, S(\theta)u, \varepsilon) \tag{1.15}$$

for all $u \in E^n$, $t \in R$, $0 < \varepsilon \leqq \varepsilon_0$, $\theta \in R^k$. Although our interest is in systems with symmetry, our analysis will depend on the properties (1.15) only and not explicitly on the group property $S(\theta_1)S(\theta_2) = S(\theta_1 + \theta_2)$. The group property merely imposes conditions on the shapes of the integral manifolds under consideration. We note that because of the first of (1.15), $S(\theta)\, e^{At} = e^{At}S(\theta)$ for all t and θ. Furthermore,

$$\begin{aligned} S(\theta)f(t, x, \varepsilon) &= S(\theta)\, e^{-At}h(t, e^{At}x, \varepsilon) \\ &= e^{-At}h(t, e^{At}S(\theta)x, \varepsilon) \end{aligned}$$

and thus, if $z = S(\theta)x$, we have

$$\dot{z} = \varepsilon f(t, z, \varepsilon)$$

so that the "standard form" is invariant under the transformation $z = S(\theta)x$.

Also, since

$$S(\theta)f(t, \xi, \varepsilon) = f(t, S(\theta)\xi, \varepsilon),$$

and

$$S(\theta)f_0(\xi) = f_0(S(\theta)\xi), \tag{1.16}$$

the averaging operator commutes with $S(\theta)$. Thus, if $\xi^0 \neq 0$ is a constant solution of the averaged equation (1.9), then, from (1.16), $S(\theta)\xi^0$ is also a constant solution for each θ. Thus, we have a k-parameter family of constant solutions forming an invariant set of the averaged system of equations.

Since

$$f_0(S(\theta)\xi^0) = 0, \tag{1.17}$$

differentiation of (1.17) with respect to the jth component of θ yields

$$\frac{\partial f_0(S(\theta)\xi^0)}{\partial \xi} \frac{\partial S(\theta)}{\partial \theta_j} \xi^0 = 0 \tag{1.18}$$

for all $\theta \in R^k$. Thus, we see that the matrix $\partial f_0(S(\theta)\xi^0)/\partial\xi$ has a zero eigenvalue for all θ including $\theta = 0$ and the corresponding eigenvector is $(\partial S(\theta)/\partial\theta_j)\xi^0$, which is tangent to the invariant surface.

We will discuss here any system of equations in which the averaged equations have invariant surfaces of constant solutions. The surface of constant solution may be due to symmetry or due to other causes as was the case in the example of the van der Pol equation. There are cases in which the symmetry parameter and the parameter corresponding to θ in the van der Pol equation coincide; then, the symmetry parameter becomes redundant.

In the standard results relating to integral manifolds and the method of averaging, it is necessary to have a global parameterization of the integral manifold. In our case this is not the case and manifold without global parameterization can also be treated by our procedures.

There are some references on persistence of invariant manifolds with hyperbolic structure [7], [8], [9] that do not have the restriction of global parameterization. These results, however, do not pertain to the problem discussed here.

This work was motivated by example B in the text. Example B was studied in [10] and [11] by Bajaj and Sethna by the method of alternate problems [6] for periodic solution. During this study it was found that the standard results of the method of averaging relating to integral manifolds were not applicable to this problem independently of which coordinate system was chosen for the analysis.

2. A Lyapunov function for the invariant surface. Our objective is to prove a theorem for a system of the form (1.7). This equation can be written in the following form by using the slow time $\tau = \varepsilon t$:

$$\xi' = f_0(\xi) + \varepsilon g(\tau, \xi, \varepsilon) \tag{2.1}$$

where $d/d\tau = '$ and $g(\tau, \xi, \varepsilon) = \tilde{g}(\tau/\varepsilon, \xi, \varepsilon)$.

We assume that the averaged system

$$\xi' = f_0(\xi) \tag{2.2}$$

has a k parameter family of constant solutions.

More precisely, we assume that $M = \{\xi \in R^n : f_0(\xi) = 0\}$ is a nonempty, bounded set such that the rank of the Jacobian $(\partial f_0/\partial \xi)(\xi_0)$ is exactly equal to $l = n - k$ at each point $\xi_0 \in M$. This assures that M is a compact, C^2-submanifold of R^n. The set M is overdetermined by the n equations $f_0(\xi) = 0$. Choose l functions from the set of n equations $f_0(\xi) = 0$ which determine M; specifically, let $F = (F^1, \cdots, F^l)$ be l functions such that $M = \{\xi: F(\xi) = 0$ and such that $(\partial F/\partial \xi)(\xi_0)$ has rank l for each $\xi_0 \in M$. Let us assume that coordinates $\xi_1, \cdots, \xi_n$ have been ordered so that the Jacobian $\{(\partial F^i/\partial \xi^j)(\xi_0)\}$, $i, j = 1, \cdots, l$ is nonsingular. Thus ordering depends on the point ξ_0, but ξ_0 will be fixed for most of the argument given below.

We assume throughout this and the next section that the linearized averaged equations

$$y' = \frac{\partial f_0(\xi_0)}{\partial \xi} y \tag{2.3}$$

have k zero eigenvalues and $l = n - k$ eigenvalues with negative real parts for each $\xi_0 \in M$. We note that this assumption is independent of coordinates, since the matrix of coefficients of the linearized equations transforms by a similarity transformation when variables are changed.

Now we shall construct a specific coordinate system which locally brings M into a plane. Introduce coordinates (ρ, θ) by

$$\begin{aligned} \rho^1 &= F^1(\xi), \\ &\vdots \\ \rho^l &= F^l(\xi), \\ \theta^1 &= \xi^{l+1}, \\ &\vdots \\ \theta^k &= \xi^n. \end{aligned} \tag{2.4}$$

The Jacobian of this change of variables is

$$(2.5)\qquad \left(\begin{array}{ccc|cccccc} \frac{\partial F^1}{\partial \xi^1} & \cdots & \frac{\partial F^1}{\partial \xi^l} & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{\partial F^1}{\partial \xi^n} \\ \vdots & & \vdots & \vdots & & & & & \\ \frac{\partial F^l}{\partial \xi^1} & \cdots & \frac{\partial F^l}{\partial \xi^l} & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{\partial F^l}{\partial \xi^n} \\ & & & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ & & & 0 & 1 & 0 & 0 & \cdots & 0 \\ & 0 & & & & \vdots & & & \vdots \\ & & & 0 & & & & \cdots & 1 \end{array}\right)$$

which is nonsingular at $\xi = \xi_0$. Thus by the inverse function theorem there exist open sets $U_1 \subset R^l$, $U_2 \subset R^k$, $W \subset R^n$ such that $(\rho, \theta) \in U_1 \times U_2$ are valid coordinates in R^n for the neighborhood W of $\xi_0 \in M \subset R^n$. In these coordinates the manifold M is given by $\{(\rho, \theta) \colon \rho = 0\}$ and the equations (2.1) are of the form

$$(2.6)\qquad \begin{aligned} \theta' &= p_1(\theta, \rho) + \varepsilon g_1(\theta, \rho, \tau, \varepsilon), \\ \rho' &= \hat{p}_2(\theta, \rho) + \varepsilon g_2(\theta, \rho, \tau, \varepsilon). \end{aligned}$$

Since M is given by $\rho = 0$ and when $\varepsilon = 0$ the set M consists of critical points, we have $p_1(\theta, 0) = \hat{p}_2(\theta, 0) \equiv 0$. Thus the Jacobian $(\partial(p_1, \hat{p}_2)/\partial(\theta, \rho))(\theta, 0)$ must be of the form

$$\begin{bmatrix} 0 & K(\theta) \\ 0 & H(\theta) \end{bmatrix}.$$

Since the eigenvalues of this Jacobian must be the same as the eigenvalues of $(\partial f_0/\partial \xi)(\xi_0)$, we see that all the eigenvalues of $H(\theta)$ must have negative real parts. Write $\hat{p}_2(\theta, \rho) = H(\theta)\rho + p_2(\theta, \rho)$ so $p_2(\theta, 0) = 0$ and $(\partial p_2/\partial \rho)(\theta, 0) = 0$; i.e., p_2 is second order in ρ. Thus in the new coordinates the equations (2.1) become

$$(2.7)\qquad \begin{aligned} \theta' &= p_1(\theta, \rho) + \varepsilon g_1(\theta, \rho, \tau, \varepsilon), \\ \rho' &= H(\theta)\rho + p_2(\theta, \rho) + \varepsilon g_2(\theta, \rho, \tau, \varepsilon). \end{aligned}$$

Although the above derivation of (2.7) is simpler, some readers may prefer an alternate derivation as given in the Appendix. The derivation given in the Appendix is based on some work of Hale and Stokes [12]. In it the coordinates are based on the solutions of (2.3) and give a clearer mental picture of the situation.

Now let us define a Lyapunov function for equations (2.7). Since all the eigenvalues of $H(\theta)$ have negative real parts, the matrix

$$D(\theta) = \int_0^\infty e^{H^T(\theta)\tau} e^{H(\theta)\tau}\, dt$$

is an $l \times l$ positive definite symmetric matrix which satisfies the identity

$$(2.8)\qquad H^T(\theta)D(\theta) + D(\theta)H(\theta) = -I.$$

By restricting their sizes, if necessary, we may assume that U_1 and U_2 are balls of radii r_1 and r_2 with centers at 0 and 0_2 respectively. Let $\alpha(\theta) = \{r_2^2 - \|\theta - 0_2\|^2\}^2$ so $\alpha \colon U_2 \to R$ is a smooth function which is positive on U_2 and $\alpha(\theta)$ and its first partial derivatives tend to zero as θ tends to the boundary of U_2. Define

$$\beta(\rho)=\begin{cases}1 & \text{if } \|\rho\|\leqq r_1/2,\\ 0 & \text{if } \|\rho\|\geqq r_1,\\ \gamma(2\rho/r_1) & \text{if } r_1/2<\|\rho\|<r_1\end{cases}$$

where $\gamma(x)$ is the cubic polynomial which has a maximum of 1 at 1 and a minimum of 0 at 2, namely $\gamma(x)=(2x-1)(x-2)^2$. Note that β is a C^1 function which is identically 1 when $\|\rho\|\leqq r_1/2$ and identically zero when $\|\rho\|>r_1$. The two functions α and β will be used to extend the local definition of our Lyapunov function to a global Lyapunov function.

Define $v(\rho,\theta)=\rho^TD(\theta)\rho\alpha(\theta)\beta(\rho)$. Clearly v is positive for $0<\|\rho\|<r_1$ and $\|\theta-0_2\|<r_2$ and $v(0,\theta)\equiv 0$. Also v tends to zero as $\|\rho\|\to r_1$ or as $\|\theta-0_2\|\to r_2$ as do the first partials of v. Thus if we consider v as a function of the original variables ξ which are valid in all of R^n, we see that v is well defined in W and v along with its first partials tends to zero as ξ tends to the boundary of W. Thus we can make a C^1 extension of v by defining v to be identically zero for $\xi\in R^n-W$.

Let us compute the derivative of v along the trajectories of (2.7) when $\varepsilon=0$ and for $\|\rho\|<r_1/2$ (so $\beta\equiv 1$). This derivative is

$$v_0'=-\rho^T\rho\alpha+\rho^T\frac{\partial D}{\partial\theta}p_1\rho\alpha+2\rho^TDp_2\alpha+\rho^TD\rho\frac{\partial\alpha}{\partial\theta}p_1. \tag{2.9}$$

Now $p_1=O(|\rho|)$ and $p_2=O(|\rho|^2)$ as $|\rho|\to 0$ and so for fixed θ

$$v_0'=-\rho^T\rho\alpha+O(|\rho|^3).$$

Thus for each θ we can choose an $\eta(\theta)$ such that $v_0'<0$ for $0<|\rho|<\eta(\theta)$ and moreover $\eta(\theta)$ can be taken as continuous in θ. Thus the set $U=\{(\rho,\theta): |\rho|<\eta(\theta)\}$ is an open neighborhood of $\rho=0$, $\theta\in U_2$.

Going back to the original variables ξ in R^n, we have constructed a globally defined C^1-function $v(\xi)$ which satisfies (1) $v(\xi)\geqq 0$ for all ξ; (2) $v(\xi)>0$ for $\xi\in V-M$; (3) $v(\xi)=0$ for $\xi\in M$; (4) $v_0'<0$ for $\xi\in W'-M$ where W' is the set of all ξ corresponding to U. In (4) the computation of the derivative is taken along the solution of (2.2).

Now M is a compact set and so if we perform the above construction for each point of M there are a finite number of coordinate systems such that the unions of the corresponding W and W' cover all of M. That is, there exist a finite number k, and globally defined C^1 functions $v_1,\cdots,v_k$, which satisfy (1), (2), (3), and (4) above on the open sets W_i and W_i and such that

$$\bigcup_i^k W_i\supset M \quad\text{and}\quad \bigcup_i^k W_i'\supset M.$$

Define $w(\xi)=v_1(\xi)+\cdots+v_k(\xi)$ and $N=(\bigcup_i^k W_i)\cap(\bigcup_i^k W_i')$. N is an open neighborhood of M in R^n and the function W is C^1 and satisfies (1) $W(\xi)\geqq 0$ for all $\xi\in R^n$; (2) $W(\xi)>0$ for all $\xi\in N-M$; (3) $W(\xi)=0$ for $\xi\in M$; (4) $W_0'(\xi)<0$ for all $\xi\in 0-M$ where W_0' is the derivative of w along the trajectories of (2.1) when $\varepsilon=0$.

3. Main results. We are now ready to prove a theorem for (1.2) using the Lyapunov function constructed in the previous section. The system in standard form is:

$$\dot{x}=\varepsilon f(t,x,\varepsilon). \tag{3.1}$$

Then as shown in the introduction, using the near identity change of variables (1.3), (3.1) can be reduced to

$$\xi' = f_0(\xi) + \varepsilon g(\tau, \xi, \varepsilon) \tag{3.2}$$

where $\tau = \varepsilon t$, $d/d\tau = '$, $g(\tau, \xi, \varepsilon) = \hat{g}(t/\varepsilon, \xi, \varepsilon)$ and where $\hat{g}$ is as in (1.8). The averaged equation corresponding to (3.2) is

$$\xi' = f_0(\xi). \tag{3.3}$$

For any set M and any positive number δ we define a δ-neighborhood of M to be

$$B_\delta(M) = \{\xi \in R^n: d(\xi, M) < \delta\},$$

where

$$d(\xi, M) = \inf\{d(\xi, \zeta): \zeta \in M\}.$$

We now state and prove our main result.

THEOREM 1. *Let the functions f_0 and g be C^2 for $|\xi| < R$, $\tau \geqq \tau_0$, $0 \leqq \varepsilon \leqq \varepsilon_2$. Assume that $M = \{\xi: f_0(\xi) = 0\}$ is nonempty, is contained in $|\xi| < R$, for some $R > 0$, and that the rank of $(\partial f_0)/(\partial \xi)(\xi_0)$ is identically equal to $l = n - k$ for all $\xi_0 \in M$. Also assume $(\partial f_0)/(\partial \xi)(\xi_0)$ has l eigenvalues with negative real parts and the remaining k eigenvalues with zero real parts for all $\xi_0 \in M$.*

Then for any $\eta > 0$ there exist a δ_η and an ε_η depending on η such that if $\psi(\tau, \xi_0, \tau_0, \varepsilon)$ is the solution of (3.2) *with $\psi(\tau_0, \xi_0, \tau_0, \varepsilon) = \xi_0 \in B_{\delta_\eta}(M)$ then $\psi(\tau, \xi_0, \tau_0, \varepsilon) \in B_\eta(M)$ for all $\tau \geqq \tau_0$ and $0 < \varepsilon < \varepsilon_\eta$.*

The proof is based on a procedure used by Malkin [13] and is similar to the one used by Sethna [1].

Proof. Let w be the Lyapunov function which was constructed in the previous section. The derivative of w along the trajectories of (3.2) is

$$w' = w_0' + \varepsilon \frac{\partial w}{\partial \xi} g$$

where w_0' is as in the previous section.

Let η be so small that the η-neighborhood of M, $B_\eta(M)$, lies inside the open neighborhood N which was constructed in the previous section. Let $a = \min\{w(\xi): d(\xi, M) = \eta\}$. Since $w(\xi) > 0$ for $\xi \in N - M$ and M is compact, we have $a > 0$. Since $w(\xi) = 0$ for $\xi \in M$ and w is continuous, there is a $\delta = \delta_\eta > 0$ such that if $\xi \in B_\delta(M)$, then $w(\xi) < a/2$.

Let C be the closure of $B_\eta(M) - B_\delta(M)$; so C is a compact set in R^n which does not meet M. Since $w_0'(\xi)$ is negative for $\xi \in C$, there is a $b > 0$ such that $w_0'(\xi) < -b$ for $\xi \in C$. Now w and g are C^1 on C and so $(\partial w/\partial \xi)g$ is bounded on C. Thus there is an $\varepsilon_\eta > 0$ such that $|\varepsilon(\partial w/\partial \xi)(\xi)g(\xi)| < b/2$ for all $0 \leqq \varepsilon \leqq \varepsilon_\eta$ and $\xi \in C$ and therefore $w'(\xi) < -b/2$ for $0 \leqq \varepsilon \leqq \varepsilon_\eta$ and $\xi \in C$.

Let $\psi(\tau) = \psi(\tau, \xi_0, \tau_0, \varepsilon)$ be the solution of (3.2) such that $\psi(\tau_0) = \xi_0 \in B_\delta(M)$. Define $w(\tau) = w(\psi(\tau))$. Since $\psi(\tau_0) = \xi_0 \in B_\delta(M)$, we have $w(\tau_0) < a/2$. Assume that for some time beyond τ_0 this solution leaves the η neighborhood of M. Let $T > \tau_0$ be the first time that it leaves, then $\psi(\tau) > B_\eta(M)$, $\tau_0 \leqq \tau < T$, and $d(\psi(T), M) = \eta$. Thus $w(T) \geqq a$. Now for each τ in the range $\tau_0 \leqq \tau < T$, either: (i) $\psi(\tau) \in B_\delta(M)$ so $w(\tau) < a/2$; or (ii) $\psi(\tau) \in B_\eta(M) - B_\delta(M) \subset C$ where $w'(\tau) < 0$. Thus $w(\tau) \leqq a/2$ for $\tau_0 \leqq \tau < T$ but $w(T) \geqq a$. This contradicts the continuity of $w(\tau)$ and the theorem is established.

Remark 1. Our proof uses several changes of variables which are not necessary to perform in applications. The main hypothesis concerns the function $f_0(\xi)$ which can be computed directly from the original equations (1.2) and hence (1.1).

Remark 2. If one or more of the eigenvalues of $(\partial f_0)/(\partial \xi)(\xi_0)$ has positive real part for some $\xi_0 \in M$, one can use an argument similar to the one in [1] to prove an instability result. Namely, there is an $\eta > 0$ such that for any $\delta > 0$ and any small ε there is a $T > \tau_0$ and a $\xi_0 \in B_\delta(M)$ such that $d(\psi(T, \xi_0, \tau_0, \varepsilon), M) \geqq \eta$.

4. A nonlocal result. In many nonlinear problems the system has more than one stable invariant set in the form of almost periodic solutions.

It then becomes a matter of importance in applications to determine which of these solutions is the one to which a given solution approaches as $t \to \infty$, i.e. to determine the nonlocal domain of attraction of these stable almost periodic solutions. An analysis leading to such a result was given in [14]. A similar result is possible in the case when more than one disjoint positively invariant manifolds exist, as is the case in the class of systems discussed here. The analysis depends on a finite time result which is Bogolyubov's first theorem [5]:

THEOREM. *If system* (3.1) *satisfies the conditions stated earlier, then given any* $\sigma > 0$, $\mu > 0$, *however small, and* $L > 0$, *however large, there exists an* ε^*, $0 < \varepsilon^* < \varepsilon_0$ *such that if* $\psi(t, \xi^0, \tau_0)$ *is a solution of* (3.3), *starting at* $|\xi^0| < R$ *is defined for* $0 \leqq t < \infty$ *and it along with its* σ *neighborhood remains in the set* $|\zeta| < R$ *for all* $t \geqq t_0$, *then for all* ε, $0 < \varepsilon \leqq \varepsilon^*$, *the solution* $x(t, x^0, t^0)$ *of* (3.1) *with* $x^0 = \xi^0$ *satisfies*

$$|x(t, x^0, t_0) - \psi(t, \xi^0, t_0)| < \mu$$

for all t, $0 \leqq t \leqq L/\varepsilon$.

Using this theorem and Theorem 1, one can prove the following:

THEOREM 2. *Under the hypotheses of Theorem* 1, *let* $x(t, x^0, t_0)$ *be a solution of* (3.1) *with* $x^0 = x(t_0, x^0, t_0)$ *and let* $\psi(t, \xi^0, t_0)$ *be a solution of* (3.3) *with* $\psi(t_0, \xi^0, t_0) = \xi^0$ *and* $x^0 = \xi$. *Furthermore suppose* $\psi(t, \xi^0, t_0)$ *is such that* $d(M, \psi(t, \xi^0, t_0) \to 0$ *as* $t \to \infty$ *and* ψ *with its* σ *neighborhood lies in the ball of radius R, then there exist some* $T > t_0$ *and* ε_η *such that for any given* η $|x(t, x^0, t_0) - \psi(t, \xi^0, t_0)| < \eta$ *for* $t_0 \leqq t \leqq T$ *and* $x(t, x_0, t_0) \in B_\eta(M)$ *for all* $t \geqq T$, *for* $0 < \varepsilon \leqq \varepsilon_\eta$.

Thus, if the solution of the averaged equation approaches M, then the solution of the original system starting with the same initial conditions is in the domain of attraction of M.

5. Applications. We give below two applications of our results. The first is a nonautonomous system of fourth order, the averaged equation of which has a one-dimensional manifold in the form of S^1. In the second example the system is of 8th order with a four-dimensional center manifold. Our analysis is done in this four-dimensional state space of the center manifold. In this space it is shown that the averaged system has, depending on the case, a one- or two-dimensional manifold which has a positively invariant set in its neighborhood. Furthermore it is possible to give explicit estimates of these positively invariant sets.

A. Spherical pendulum with vertically oscillating support. Consider a spherical pendulum suspended at the top and the point of suspension oscillating vertically with a dimensionless velocity $\mu(t)$.

The equations of motion, in dimensionless form, up through cubic terms, can be shown to be

$$\ddot{\xi} + \hat{c}\dot{\xi} + (w_0^2 - \dot{\mu}(t))\xi = -\tfrac{1}{2}(w_0^2 - \dot{\mu}(t))\xi(\xi^2 + \eta^2) - \xi(\dot{\xi}^2 + \dot{\eta}^2) - \xi(\xi\ddot{\xi} + \eta\ddot{\eta}), \tag{5.1}$$

$$\ddot{\eta} + \hat{c}\dot{\eta} + (w_0^2 - \dot{\mu}(t))\eta = -\tfrac{1}{2}(w_0^2 - \dot{\mu}(t))\eta(\xi^2 + \eta^2) - \eta(\dot{\xi}^2 + \dot{\eta}^2)\eta(\xi\ddot{\xi} + \eta\ddot{\eta}), \tag{5.2}$$

where $w_0^2 = g/l$ is the linear natural frequency, $\hat{c}$ is the damping coefficient and the coordinates ξ and η are the nondimensional projections of the displacement on two orthogonal vertical planes.

This problem has been studied by Sethna and Hemp [15] under slightly different assumptions. We note here that the dissipation terms in (5.1) and (5.2) provide a moment about the vertical axis in contrast to the case in [15] and thus the system is not reducible to a second order system.

We are interested in studying small nonlinear motions when primary parametric resonance occurs, that is, when $\mu(t)$ is sinusoidal and its frequency ν is nearly twice the natural frequency. To make the above conditions explicit, let $\xi=\varepsilon^{1/2}\alpha$, $\eta=\varepsilon^{1/2}\beta$, $\tau=w_0 t$, $\gamma=\nu/w_0=2(1-\varepsilon d)$, $\hat{c}=\varepsilon w_0 c$ and $\mu(t)=\varepsilon\nu\sin\nu t$ where d is the "detuning" parameter and ε is a small parameter. Furthermore let

$$\alpha=u_1,\quad \dot{\alpha}=u_2,\quad \beta=u_3,\quad \dot{\beta}=u_4. \tag{5.3}$$

Using (5.3), the equations (5.1), (5.2) transform to

$$\dot{u}=Au+\varepsilon h(\tau,u,\varepsilon) \tag{5.4}$$

where

$$A=\left[\begin{array}{cc|cc} 0 & \gamma/2 & 0 & 0 \\ -\gamma/2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \gamma/2 \\ 0 & 0 & -\gamma/2 & 0 \end{array}\right] \tag{5.5}$$

and

$$\mathbf{h}(\tau,\mathbf{u},\varepsilon)\equiv\begin{bmatrix} du_2 \\ (\nu^2\cos\nu\tau)u_1-cu_2-du_1+\frac{1}{2}u_1(u_1^2+u_3^2)-u_1(u_2^2+u_4^2)+O(\varepsilon) \\ du_4 \\ (\nu^2\cos\nu\tau)u_3-cu_4-du_3+\frac{1}{2}u_3(u_1^2+u_3^2)-u_3(u_2^2+u_4^2)+O(\varepsilon) \end{bmatrix}. \tag{5.6}$$

The physical system is invariant to rotations about the vertical axis through the point of suspension and this is reflected in that the equations (5.6) are covariant to the rotation matrix

$$\mathbf{S}(\theta)=\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{bmatrix},\qquad \theta\in[0,2\pi], \tag{5.7}$$

that is, the matrix $\mathbf{A}$ and the function ν satisfy the conditions (1.15).

Equations (5.4) are reduced to the standard form

$$\dot{x}=\varepsilon f(\tau,x,\varepsilon) \tag{5.8}$$

by the transformation $u=e^{A\tau}x$. Note that the right-hand side is $4\pi/\gamma$ periodic in τ.

The averaged equation corresponding to (5.8) is

$$\dot{\xi}=\varepsilon f_0(\xi) \tag{5.9}$$

where

$$f_0(\xi) \equiv \begin{bmatrix} -\frac{1}{2}c\xi_1 + d\xi_2 + (\xi_2 - \xi_1) - \frac{1}{16}\{6\xi_1(\xi_1\xi_2 + \xi_3\xi_4) \\ + \xi_2(\xi_2^2 + \xi_4^2) - 5\xi_2(\xi_1^2 + \xi_3^2)\} \\ -\frac{1}{2}c\xi_2 - d\xi_1 + (\xi_1 + \xi_2) + \frac{1}{16}\{6\xi_2(\xi_1\xi_2 + \xi_3\xi_4) \\ + \xi_1(\xi_1^2 + \xi_3^2) - 5\xi_1(\xi_2^2 + \xi_4^2)\} \\ -\frac{1}{2}c\xi_3 + d\xi_4 + (\xi_4 - \xi_3) - \frac{1}{16}\{6\xi_3(\xi_1\xi_2 + \xi_3\xi_4) \\ + \xi_4(\xi_2^2 + \xi_4^2) - 5\xi_4(\xi_1^2 + \xi_3^2)\} \\ -\frac{1}{2}c\xi_4 - d\xi_3 + (\xi_3 + \xi_4) + \frac{1}{16}\{6\xi_4(\xi_1\xi_2 + \xi_3\xi_4) \\ + \xi_3(\xi_1^2 + \xi_3^2) - 5\xi_3(\xi_2^2 + \xi_4^2)\} \end{bmatrix}. \tag{5.10}$$

After some lengthy calculations, it can be shown that, in terms of the polar variables a_1, a_2, ψ_1 and ψ_2 defined through

$$\begin{aligned} \xi_1 &= a_1 \cos \psi_1, \qquad \xi_2 = a_1 \sin \psi_1, \\ \xi_3 &= a_2 \cos \psi_2, \qquad \xi_4 = a_2 \sin \psi_2, \end{aligned} \tag{5.11}$$

the only nontrivial constant solutions of (5.9) are given by

$$\psi_1 = \psi_2 = \frac{1}{2}\cos^{-1}\left(\frac{d - r - c/2}{2}\right), \tag{5.12a}$$

and

$$a_1^2 + a_2^2 = 16r^2 \tag{5.12b}$$

where r^2 satisfies the quadratic

$$r^4 - 2\,dr^2 + \left(\frac{c}{2}\right)^2 + d^2 - 2 = 0 \tag{5.12c}$$

which has one solution

$$r_1^2 = d + \tfrac{1}{2}\sqrt{8 - c^2} \tag{5.12d}$$

if $4d^2 > 8 - c^2$ and two solutions, one as in (5.12d) and

$$r_2^2 = d - \tfrac{1}{2}\sqrt{8 - c^2} \quad \text{if } 4d^2 < 8 - c^2. \tag{5.12e}$$

Each of the manifolds is a circle in R^4. This can be seen as follows. Equation (5.12b) in view of (5.11) represents S^3 in R^4. Furthermore from (5.11) we have $\xi_1/\xi_2 = \coth \psi_1$ and $\xi_3/\xi_4 = \coth \psi_2$. Since ψ_1 and ψ_2 are constants, each of these equations represents a three-dimensional plane through the origin and their intersection is a two-dimensional plane through the origin. The manifold is the intersection of S^3 and this two-dimensional plane and is therefore a circle.

In addition to the above manifolds there is a zero solution of the averaged system (5.9). This solution can be shown to be asymptotically stable for $4d^2 > 8 - c^2$ and unstable when $4d^2 < 8 - c^2$. It then follows from the classical result that in the neighborhood of the origin there is a stable or unstable periodic solution respectively. Not surprisingly, the condition $4d^2 > 8 - c^2$ is also the condition for the existence of two manifolds M_1 and M_2 and $4d^2 < 8 - c^2$ is the condition for the existence of only one manifold M_1. Furthermore, it can be shown that M_1 is stable and M_2 is unstable.

Then based on Theorem 1 and the classical theorem we can draw the following conclusions about the motion. If $4d^2 > 8 - c^2$ the straight vertical portion is stable. The motions corresponding to M_1 and M_2 are motions in some vertical planes. The one corresponding to M_1 is of larger amplitude and stable and the one corresponding to M_2 is of smaller amplitude and unstable. The motion that will actually occur will of course depend on the domains of attraction of the vertical position and that of the invariant set corresponding to M_1. If the averaged equations are integrated numerically, then by appealing to Theorem 2 one can draw conclusions about these domains of attractions.

If $4d^2 < 8 - c^2$ we have the vertical position unstable and the system performs stable periodic motions in some vertical plane.

B. Flow-induced oscillations in articulated tubes. Consider a two-segment articulated tubes system hanging vertically. The fluid enters the tubes at the top and is discharged at the bottom end of the lower tube tangentially. A cartesian coordinate system is fixed at the top and the Z-axis coincides with the downward vertical position. We assume that the fluid is incompressible, the velocity profile at any cross-section is uniform and the velocity of the fluid relative to the tubes is constant. Both the tubes have the same circular cross-section and the diameter of each tube is small compared to its length. We further assume that the bending stiffnesses of the joints have radial symmetry and the elastic restoring forces are linearly proportional to the angles between the center lines of adjacent tubes. This system has rotational symmetry about the vertical axis. Periodic motion of such systems has been studied by Bajaj and Sethna [10] by another method in great detail. We therefore suppress many of the details of the calculations and merely give an outline of the essential features of the method of analysis. The coordinate system is as follows: x_{11} and $x_{11}+x_{12}$ are, respectively, the nondimensional position coordinates along the X-axis of the end points of the upper and lower tube segments. Similarly x_{21} and $x_{21}+x_{22}$ are, respectively, the position coordinates along the Y-axis. The XY plane is perpendicular to the X-axis.

The system depends on five dimensionless parameters a, k, β, ρ and G. Parameter "a" is the ratio of the lengths of upper and lower segments. "k" is the ratio of stiffness of the upper joint to that of the lower joint. "β" is the ratio of mass of fluid to the total mass at any instant. The flow rate in dimensionless form is represented by ρ and G is the dimensional gravity parameter.

Let

$$Z = (x_{11}, x_{12}, \dot{x}_{11}, \dot{x}_{12}, x_{21}, x_{22}, \dot{x}_{21}, \dot{x}_{22})^{\tau}. \tag{5.13}$$

The system of equations then take the form

$$\dot{Z} = A(\rho)Z + g(Z, \rho) \tag{5.14}$$

where we have suppressed the dependence on other parameters and where $Z \in R^8$ and g is an odd function in Z that is very complicated and is given in [10].

The physical system is invariant to rotations about the Z-axis. It can be easily shown that the matrix A and function g therefore satisfy

$$\tilde{\tilde{S}}(\theta)A = A\tilde{\tilde{S}}(\theta) \tag{5.15}$$

and

$$\tilde{\tilde{S}}(\theta)g(z, \rho) = g(\tilde{\tilde{s}}(\theta)z, \rho)$$

where $\tilde{\tilde{S}}(\theta)$ is a one-parameter matrix given by

$$\tilde{\tilde{S}}(\theta) = \begin{bmatrix} \cos\theta I_4 & \sin\theta I_4 \\ -\sin\theta I_4 & \cos\theta I_4 \end{bmatrix}, \qquad \theta \in [0, 2\pi].$$

We are interested in bifurcation phenomena when the equilibrium position $Z=0$ gets unstable as the flow rate ρ is increased. It is shown in [10] that for small values of ρ, all the eigenvalues of matrix **A** are in the left half of the complex plane. As ρ is increased, it reaches a critical value, $\rho=\rho_{cr}$ when a double pair of complex conjugate eigenvalues cross the imaginary axis rendering $Z=0$ unstable.

Letting $\rho=\rho_{cr}+\mu$, (5.14) can be written as

$$\mathring{z}=\bar{A}_0 z+\mu\bar{A}_1(\mu)z+\bar{g}(z,\mu) \tag{5.16}$$

where $\bar{A}_0$ has two pure imaginary pairs of eigenvalues $\pm iw_{01}$, $\pm iw_0$ and the other four eigenvalues are in the left half of the complex plane. For small enough μ and $|z|$, the center manifold theorem [3] guarantees the existence of a four-dimensional center manifold on which the system dynamics is governed by equations of the form

$$\dot{y}=Ay+\mu A_1(\mu)y+k(y,\mu) \tag{5.17}$$

where $y\in R^4$ and the function **k** is as given below.

Let $y=\mu^{1/2}u$ for $0<\mu\ll 1$, (the case of $\mu<0$ can be similarly treated) so that (5.17) now takes the form

$$\dot{u}=Au+\mu A_1(\mu)u+\mu\bar{k}(u,\mu)$$

or

$$\dot{u}=Au+\mu h(u,\mu) \tag{5.18}$$

where

$$A=\left[\begin{array}{cc|cc} 0 & w_0 & & \\ -w_0 & 0 & \multicolumn{2}{c}{0} \\ \hline \multicolumn{2}{c|}{0} & 0 & w_0 \\ & & -w_0 & 0 \end{array}\right],$$

$$A_1(0)=\left[\begin{array}{cc|cc} \eta & \tilde{w} & & \\ -\tilde{w} & \eta & & \\ \hline & & \eta & \tilde{w} \\ & & -\tilde{w} & \eta \end{array}\right],$$

and

$$\bar{k}(u,0)=\begin{bmatrix} (u_1^2+u_3^2)(B_1u_1+B_2u_2)+(u_1u_2+u_3u_4)(B_3u_1+B_4u_2) \\ \qquad +(u_2^2+u_4^2)(B_5u_1+B_6u_2) \\ (u_1^2+u_3^2)(B_7u_1+B_8u_2)+(u_1u_2+u_3u_4)(B_9u_1+B_{10}u_2) \\ \qquad +(u_2^2+u_4^4)(B_{11}u_1+B_{12}u_2) \\ (u_1^2+u_3^2)(B_1u_3+B_2u_4)+(u_1u_2+u_3u_4)(B_3u_3+B_4u_4) \\ \qquad +(u_2^2+u_4^2)(B_5u_3+B_6u_4) \\ (u_1^2+u_3^2)(B_7u_3+B_8u_4)+(u_1u_2+u_3u_4)(B_9u_3+B_{10}u_4) \\ \qquad +(u_2^2+u_4^2)(B_{11}u_3+B_{12}u_4) \end{bmatrix}.$$

The constant $\eta+i\tilde{w}$ is the rate of change with ρ of the critical eigenvalue. The constants B_i, $i=1,2,\cdots,12$ are determined by the nonlinear terms.

The system (5.18) is in the form of the autonomous systems discussed in § 1. These equations on the center manifold inherit the symmetry of the original system as shown by Ruelle [16]. Thus (5.18) satisfies (1.15) with respect to a one-parameter matrix $S(\theta)$

which can easily be shown to be the same as the matrix $S(\theta)$ of the previous example given in (5.8).

Equation (5.18) is transformed to the "standard" form

$$x' = \mu f(\tau, x, \mu) \tag{5.19}$$

by the transformation $u = e^{A\tau}x$ where $\tau = w_0 t$. The right-hand side is 2π-periodic in τ. The averaged system is then

$$\xi' = \mu f_0(\xi) \tag{5.20}$$

where

$$\mathbf{f}_0(\boldsymbol{\xi}) \equiv \frac{1}{w_0}\begin{bmatrix} \eta\xi_1 + \tilde{w}\xi_2 + \frac{1}{8}[(\xi_1^2+\xi_3^2)\{H_1\xi_1 - (H_2+2H_{10})\xi_2\} + (\xi_2^2+\xi_4^2) \\ \{(H_1-2H_9)\xi_1 - H_2\xi_2\} + 2(\xi_1\xi_2+\xi_3\xi_4)\{H_{10}\xi_1 + H_9\xi_2\}] \\ -\tilde{w}\xi_1 + \eta\xi_2 + \frac{1}{8}[(\xi_1^2+\xi_3^2)\{H_2\xi_1 + (H_1-2H_9)\xi_2\} + (\xi_2^2+\xi_4^2) \\ \{(H_2+2H_{10})\xi_1 + H_1\xi_2\} + 2(\xi_1\xi_2+\xi_3\xi_4)\{H_9\xi_1 - H_{10}\xi_2\}] \\ \eta\xi_3 + \tilde{w}\xi_4 + \frac{1}{8}[(\xi_1^2+\xi_3^2)\{H_1\xi_3 - (H_2+2H_{10})\xi_4\} + (\xi_2^2+\xi_4^2) \\ \{(H_1-2H_9)\xi_3 - H_2\xi_4\} + 2(\xi_1\xi_2+\xi_3\xi_4)\{H_{10}\xi_3 + H_9\xi_2\}] \\ -\tilde{w}\xi_3 + \eta\xi_4 + \frac{1}{8}[(\xi_1^2+\xi_3^2)\{H_2\xi_3 + (H_1-2H_9)\xi_4\} + (\xi_2^2+\xi_4^2) \\ \{(H_2+2H_{10})\xi_3 + H_1\xi_4\} + 2(\xi_1\xi_2+\xi_3\xi_4)\{H_9\xi_3 - H_{10}\xi_4\}] \end{bmatrix}$$

and where H_1, H_2, H_9 and H_{10} are some combinations of the constants B_i.

The constant solutions of the averaged system (5.20) can be obtained after some long calculations. It can be shown that the system has only three constant solutions:

$$\text{(i)} \quad \xi = a[-\sin\phi, \cos\phi, \cos\phi, \sin\phi]^T, \qquad \phi \in [0, 2\pi), \quad a^2 = -4\eta/(H_1 - H_9), \tag{5.21}$$

$$\text{(ii)} \quad \xi = a[\sin\phi, -\cos\phi, \cos\phi, \sin\phi]^T, \qquad \phi \in [0, 2\pi), \quad a^2 = -4\eta/(H_1 - H_9), \tag{5.22}$$

and

$$\text{(iii)} \quad \boldsymbol{\xi} = a[\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta\cos\phi, \sin\theta\sin\phi]^T, \quad a^2 = -8\eta/H_1, \qquad \theta, \phi \in [0, 2\pi). \tag{5.23}$$

Each of the solutions (5.21) and (5.22) is a one-parameter family, that is, they are one-dimensional manifolds M_1 and M_2, respectively, on the center manifold of the original system (5.16) and are parameterized by the variable ϕ. It can be easily checked that physically they are the same solutions and represent motions such that each point on the tube makes a circular path about the vertical. The parameter ϕ is just the arbitrary phase of the autonomous system.

The matrix $\partial f_0/\partial\xi$ has only one zero eigenvalue in these cases. This is because the motions are circular and the symmetry parameter and the arbitrary phase of the autonomous system are indistinguishable. This particular motion can be studied by standard results of the method of averaging when extended to the study of integral manifolds [6].

The constant solution (5.23) is a two-dimensional manifold, M_3 on the center manifold. This manifold is parameterized by two variables $\theta \in [0, 2\pi]$ and $\phi \in [0, 2\pi]$. The variational matrix $\partial f_0/\partial\xi$ corresponding to the solution M_3 has two zero eigenvalues. If the remaining two eigenvalues are not on the imaginary axis, we can again, using Theorem 1, conclude the existence of an invariant set which for small enough μ remains close to the manifold M_3.

The motions corresponding to M_3 can be interpreted as motions in vertical planes. The stability of the manifold M_1, M_2, M_3 can be studied for different values of the parameters. In [11] is given the variety of cases that can occur in this system.

Appendix. In this Appendix we give an alternate derivation of (2.7) based on the work of Hale and Stokes [12].

Consider (2.1),

$$\xi' = f_0(\xi) + \varepsilon g(\tau, \xi, \varepsilon). \tag{A.1}$$

We will assume that the system

$$\xi' = f_0(\xi) \tag{A.2}$$

has a k-dimensional family of constant solutions.

More precisely, we assume that the set $M = \{\xi \in R^n : f_0(\xi) = 0\}$ is a compact, C^2 manifold (submanifold of R^n) of dimension k. (A k-dimensional manifold is simply a set which can be locally parameterized by coordinates in R^k.)

A parameterization of M is a smooth map $\xi^0 : U \to R^n$ where U is an open set in R^k such that $\xi^0(\theta) \in M$ and the Jacobian $(\partial \xi^0/\partial\theta)(\theta_0)$ has maximal rank k for all $\theta_0 \in U$. Here we consider θ as a coordinate in $U \subset R^k$.

We will assume that the variational system

$$y' = \frac{\partial f_0(\xi^0(\theta))}{\partial \xi} y \tag{A.3}$$

has k zero eigenvalues and we will further assume that the corresponding eigenvectors are linearly independent so that the $n \times k$ matrix $\partial\xi^0(\theta)/\partial\theta$ has rank k. Suppose the remaining eigenvalues of (2.3) are $\lambda_1(\theta), \lambda_2(\theta), \cdots, \lambda_l(\theta)$ have nonzero real parts (here $k + l = n$). Then (A.3) has the fundamental matrix solution $Y(t) = (\partial\xi^0(\theta)/\partial\theta, Q(\theta)\, e^{H(\theta)\tau})$ where $Q(\theta)$ and $H(\theta)$ are respectively $n \times l$ and $\alpha \times l$ constant matrices dependent on θ. Every solution of (A.3) can then be written in the form $y = (\partial\xi^0(\theta)/\partial\theta)v + Q(\theta)\, e^{H(\theta)\tau} w_1$ where v and w_1 are respectively k and l constant vectors. Now let $w = e^{H(\theta)\tau} w_1$ then since $dw_1/d\tau = 0$

$$\frac{dw}{d\tau} = H(\theta) w. \tag{A.4}$$

We plan to use a coordinate system mounted on $\xi = \xi^0(\theta)$ with coordinates (θ, w).

But first we obtain an expression relating $(\partial f_0/\partial\xi)(\xi^0(\theta))$, $Q(\theta)$ and $H(\theta)$ for later reference.

Let

$$y = \frac{\partial\xi^0(\theta)}{\partial\theta} v + Q(\theta) w; \tag{A.5}$$

then

$$\frac{dy}{d\tau} = \frac{\partial\xi^0(\theta)}{\partial\theta}\frac{dv}{d\tau} + Q(\theta)\frac{dw}{d\tau}.$$

Substituting from (A.3) to (A.5) on the left side and from (2.4) on the right side, since $dv/d\tau = 0$, we have

$$\frac{\partial f_0}{\partial\xi}(\xi^0(\theta))\left[\frac{\partial\xi^0(\theta)}{\partial\theta} v + Q(\theta) w\right] = Q(\theta) H(\theta) w.$$

But $(\partial f_0)/(\partial \xi)(\xi^0(\theta))\,\partial \xi^0(\theta)/\partial\theta = 0$ and thus we have for $w \neq 0$

$$\frac{\partial f_0(\xi^0(\theta))}{\partial \xi} Q(\theta) = Q(\theta) H(\theta). \tag{A.6}$$

Now we introduce the coordinate system

$$\xi = \xi^0(\theta) + Q(\theta)\dot{\rho} \tag{A.7}$$

where ρ is a l vector, with $|\rho| < \rho_0$ when ρ_0 is some number. Since the columns of Q are linearly independent and the rank of $\partial \xi^0(\theta)/\partial\theta$ is k, the Jacobian at $\rho = 0$, $\det[\partial \xi^0(\theta)/\partial \theta, Q(\theta)] \neq 0$ and thus (A.7) is one-to-one. Substituting (A.7) into (A.1), we have

$$\begin{aligned} \frac{d\xi}{d\tau} &= \left[\frac{\partial \xi^0}{\partial \theta} + \frac{\partial Q\rho}{\partial \theta}\right]\theta' + Q(\theta)\rho' \\ &= f_0(\xi^0 + Q\rho) + \varepsilon \tilde{g}(\tau, \xi + Q\rho, \varepsilon) \end{aligned} \tag{A.8}$$

where we use the notation

$$\frac{\partial Q}{\partial \theta}\rho\theta' \equiv \sum_{j=1}^{N}\sum_{r=1}^{k} \frac{\partial Q_{ij}}{\partial \theta_r}\rho_j \frac{d\theta_r}{d\tau}.$$

Now taking into account (A.6), (A.8) can be written in the form

$$\left[\left(\frac{\partial \xi^0}{\partial \theta} + \frac{\partial Q}{\partial \theta}\rho\right), Q\right]\begin{pmatrix} \theta' \\ \rho' - H\rho \end{pmatrix} = f_0(\xi^0 + Q\rho) - \frac{\partial f_0}{\partial \xi} Q\rho + \varepsilon \tilde{g}(\tau, \xi^0 + Q\rho, \varepsilon),$$

which can be written as

$$\begin{aligned} \theta' &= p_1(\theta, \rho) + \varepsilon q_1(\theta, \rho, \tau, \varepsilon), \\ \rho' &= H(\theta)\rho + p_2(\theta, \rho) + \varepsilon q_2(\theta, \rho, \tau, \varepsilon) \end{aligned} \tag{A.9}$$

where

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = A(\theta, \rho)\left[f_0(\xi^0(\theta) + Q(\theta)\rho) - \frac{\partial f_0(\xi^0(\theta))}{\partial \xi} Q(\theta)\rho \right]$$

and

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = A(\theta, \rho)\tilde{g}(\tau, \xi^0(\theta) + Q(\theta)\rho, \varepsilon)$$

where

$$A(\theta, \rho) = \left[\frac{\partial \xi^0(\theta)}{\partial \theta} + \frac{\partial Q(\theta)}{\partial \theta}\rho, Q(\theta)\right]^{-1}.$$

Equations (A.9) are the local representation of equations (A.1) and are valid in a neighborhood of $\rho^0(\theta) \subset M$ in R^n.

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