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AVERAGING AND BIFURCATIONS IN SYMMETRIC SYSTEMS*

JACQUES HENRARD† AND KENNETH R. MEYER‡

Abstract. This paper considers a periodic system of nonlinear ordinary differential equations which admit a special symmetry. This symmetry is found in such classic equations as Duffing's equation and van der Pol's equation. It is shown that such equations can be transformed to a nonsymmetric periodic system with half the period. Special normal forms for these symmetric systems are developed as an aid to the study of the bifurcations of periodic solutions. These normal forms are derived by the use of Lie transforms.

The first author developed a computer program which explicitly preforms the normalization. This program was used to explicitly normalize many systems of equations and a sampling of these systems is discussed.

1. Introduction. Upon reviewing the list of worked examples in nonlinear oscillations one is struck by the large percentage of equations exhibiting special symmetry properties. One might be led to believe that in nature restoring forces are always odd, forcing functions are always odd harmonic, the ratio of forcing to natural frequencies is always $p:q$ where p and q are odd, etc. However a deeper look into the theory suggests that these symmetry conditions are imposed by the authors to simplify the computations. By choosing the symmetry conditions carefully, interesting phenomena can be gleaned from the first average of the equations. Since the computation of the second or third average of an equation is considerably more difficult than the computation of the first average, the imposed symmetry conditions greatly reduce the necessary work.

In order to obtain asymptotic solutions for the equations of celestial mechanics, the first author with Deprit and Rom has developed two essential tools which facilitate the computation of the average of a system of equations to a higher order. The first, the method of Lie transforms [1], [4], is an algorithm which computes the average of a system of equations by recursive formulas. The second, PSP [6], is a package of general computer subroutines which perform literal arithmetic and analytic operations on Poisson series. (Poisson series are slight generalizations of those series obtained when an analytic system of differential equations is written in polar coordinates.) These two tools have been used to construct a program which will average equations of the form $\ddot{x}+\omega^2x=F(x, \dot{x}, A\cos t, B\sin t, \varepsilon)$ to high order in the literal sense. The details of this program will be given below. Since the arduous computations of the method of averaging can now be done by a computer, it seems appropriate to analyze the role of symmetries in averaging. As a starting point we consider a periodic system of equations which admits the type of symmetry found in Duffing's equation. It is shown that systems with this symmetry can be transformed into another periodic system with half the period and without symmetries. Thus from a theoretical point of view the symmetries add nothing new.

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† Facultés Universitaires Notre-Dame de la Paix, Namur, Belgium and Department of Mathematics, University of Cincinnati, Cincinnati, Ohio 45221.

‡ Department of Mathematics, University of Cincinnati, Cincinnati, Ohio 45221. The research of this author was supported by the National Science Foundation under Grant GP-37620.

2. General results. Consider the system

$$\dot{x} = f(x, t), \tag{2.1}$$

where $\dot{} = d/dt$, $x \in R^{2n}$ and $f: O \times R \to R^{2n}$, O open in R^{2n}, is smooth. The symmetry condition we shall impose on f is

$$f(-x, t+\pi) = -f(x, t), \qquad (x, t) \in O \times R. \tag{2.2}$$

When (2.2) holds, (2.1) is clearly 2π-periodic in t. Such a system can be obtained by writing $\ddot{\xi} + f(\xi)\dot{\xi} + g(\xi) = p(t)$, f even, g odd and p odd harmonic, as a system of two first order equations in the usual way. Thus the general remarks made below apply to the celebrated equations of Duffing and van der Pol.

A periodic solution ϕ of (2.1) of (least) period $2m\pi$ will be called odd harmonic if $\phi(t+m\pi) = -\phi(t)$. Such a periodic solution reflects the symmetry property (2.2) of the equation.

We shall show that an equation of the form (2.1) satisfying (2.2) can be transformed into a periodic system of period π. Let

$$J = \begin{pmatrix} O_n & I_n \\ -I_n & O_n \end{pmatrix}$$

be the usual $2n \times 2n$ skew symmetric matrix of Hamiltonian mechanics. Let

$$x = e^{Jt}u \tag{2.3}$$

so that (2.1) becomes

$$\dot{u} = h(u, t), \tag{2.4}$$

where $h(u, t) = -Ju + e^{-Jt}f(e^{Jt}u, t)$. Since $e^{J(t+\pi)} = -e^{Jt}$ for all t, one sees that if f satisfies (2.2), then h satisfies

$$h(u, t+\pi) = h(u, t), \tag{2.5}$$

i.e., h is π periodic. Conversely if one makes the substitution (2.3) in equation (2.4), one obtains an equation of the form (2.1) with $f(x, t) = Jx + e^{Jt}h(e^{-Jt}x, t)$. So if h satisfies (2.5), then f satisfies (2.2).

Let ϕ be a periodic solution of (2.1) of least period $2m\pi$ and define $\psi(t) = e^{-Jt}\phi(t)$ so ψ is a $2m\pi$-periodic solution of (2.4). If m is odd and ϕ is odd harmonic, then $\psi(t+m\pi) = e^{-J(t+m\pi)}\phi(t+m\pi) = (-e^{Jt})(-\phi(t)) = \psi(t)$, and so ψ is $m\pi$-periodic. Conversely if m is odd and ψ is $m\pi$-periodic, then ϕ is $2m\pi$-periodic and odd harmonic since $\phi(t+m\pi) = e^{J(t+m\pi)}\psi(t+m\pi) = (-e^{Jt})\psi(t) = -\phi(t)$. When m is even ϕ is odd harmonic if and only if ψ is odd harmonic. In summary:

THEOREM 1. *Let the change of variables* (2.3) *transform* (2.1) *to* (2.4). *Then* (2.2) *holds if and only if* (2.5) *holds. Let the transformation* (2.3) *carry a* $2m\pi$*-periodic solution* ϕ *of* (2.1) *into a* $2m\pi$*-periodic solution* ψ *of* (2.4). *When m is odd,* ϕ *is odd harmonic if and only if* ψ *has period* $m\pi$. *When m is even,* ϕ *is odd harmonic if and only if* ψ *is odd harmonic.*

Since e^{Jt} is symplectic, equation (2.1) is Hamiltonian if and only if equation (2.4) is Hamiltonian.

We are interested in the bifurcation of odd harmonic periodic solutions and so we must discuss the variational equation and characteristic multipliers. Let ϕ be an odd harmonic periodic solution of (2.1) of period 2π and let $v = x - \phi(t)$ so that (2.1) becomes

$$\dot{v} = g(v, t), \tag{2.6}$$

where $g(v, t) = f(v + \phi(t), t) - f(\phi(t), t)$. Clearly $g(0, t) = 0$ and $g(-v, t+\pi) = -g(v, t)$. Thus the variational equation of (2.1) about an odd harmonic periodic solution also satisfies condition (2.2) and so there is no loss in generality in assuming that $f(0, t) = 0$. This being the case, the linearized variational equation is

$$\dot{x} = A(t)x, \tag{2.7}$$

where $A(t) = (\partial f/\partial x)(0, t)$ and A is π-periodic. Let $X(t)$ be the fundamental matrix solution of (2.7) which satisfies $X(0) = I$. Since the original equation is 2π-periodic, the characteristic multipliers of the zero solution are the eigenvalues of $X(2\pi)$ by the usual definition. However we would like to propose that the eigenvalues of $-X(\pi)$ be called the characteristic multipliers of the zero solution of (2.1) for the following reason. When (2.3) is used to transform (2.1) to (2.4) the variational equation of (2.4) about the zero solution is

$$\dot{u} = B(t)u, \tag{2.8}$$

where $B(t) = (\partial h/\partial u)(0, t)$ and the fundamental matrix solution Y of (2.8) which satisfies $Y(0) = I$ is $Y(t) = e^{-Jt}X(t)$. Since (2.4) is π-periodic, the characteristic multipliers of the zero solution for (2.4) are the eigenvalues of $Y(\pi) = -X(\pi)$. For example, consider the two systems

$$\dot{x}_1 = (\tfrac{1}{3})x_2, \qquad \dot{x}_2 = (-\tfrac{1}{3})x_1, \tag{2.9}$$

$$\dot{x}_1 = (\tfrac{2}{3})x_2, \qquad \dot{x}_2 = (-\tfrac{2}{3})x_1, \tag{2.10}$$

taken as variational equations of a system of the form (2.1) with (2.2) and $f(0, t) = 0$. For both (2.9) and (2.10) the eigenvalues of $X(2\pi)$ are $e^{\pm i2\pi/3}$ which are cube roots of unity. However, for (2.9) the eigenvalues of $-X(\pi)$ are $-e^{\pm i\pi/3} = e^{\mp i2\pi/3}$ which are cube roots of unity and for (2.10) the eigenvalues of $X(\pi)$ are $-e^{\pm i2\pi/3} = e^{\mp i\pi/3}$ which are sixth roots of unity. Thus the eigenvalues of $-X(\pi)$ distinguish (2.9) and (2.10) whereas the eigenvalues of $X(2\pi)$ do not. Loud's investigations in [8] also seem to justify this convention.

3. Averaging. In this section we shall use the method of Lie transforms [4] to discuss the theory and implementation of averaging on a system of the form (2.1) satisfying (2.2). The process of averaging is based on constructing a change of variables $x = \phi(y, t)$ for the system (2.1), and in order to preserve (2.2) we shall require that $\phi(-y, t+\pi) = -\phi(y, t)$. This will also insure that odd harmonic periodic solutions are carried into odd harmonic periodic solutions. The class of equations considered here is very restricted since the computations are to be performed on a computer. However, the examples discussed in § 4 show that the class contains a wealth of interesting special cases.

Consider

$$\ddot{x} + \omega^2 x = \varepsilon F(x, \dot{x}, t, \varepsilon), \tag{3.1}$$

where F is a polynomial in x, $\dot{x}$, $\cos t$, $\sin t$ and ε. When (3.1) is written as a system in the usual way, the system will possess the symmetry discussed in the previous section if

$$F(-x, -\dot{x}, t+\pi, \varepsilon) = -F(x, \dot{x}, t, \varepsilon). \tag{3.2}$$

With the introduction of polar coordinates by $x = r\cos\phi$, $\dot{x} = -\omega r \sin\phi$ the equation (3.1) becomes

$$\begin{aligned} \dot{r} &= \varepsilon R(r, \phi, t, \varepsilon), \\ \dot{\phi} &= \omega + \varepsilon \Phi(r, \phi, t, \varepsilon), \end{aligned} \tag{3.3}$$

where R and $r\Phi$ are polynomials in r and ε with coefficients which are finite Fourier series in ϕ and t. Condition (3.2) on F implies

$$\begin{aligned} R(r, \phi+\pi, t+\pi, \varepsilon) &= R(r, \phi, t, \varepsilon), \\ \Phi(r, \phi+\pi, t+\pi, \varepsilon) &= \Phi(r, \phi, t, \varepsilon). \end{aligned} \tag{3.4}$$

Since we shall introduce many functions satisfying the above condition we shall call such functions π-periodic in (θ, τ). If (3.3) satisfies (3.4), then a solution $r(t)$, $\phi(t)$ is odd harmonic of period $2m\pi$ if $r(t+m\pi) = r(t)$ and $\phi(t+m\pi) = -\phi(t)$. In order to summarize the method of Lie transforms and apply it to (3.4), it is convenient to make (3.4) autonomous by introducing the new angular coordinate τ and augmenting (3.4) by $\dot{\tau} = 1$. Thus (3.4) becomes

$$\dot{z} = Z_*(z, \varepsilon), \tag{3.5}$$

where $z^T = (r, \phi, \tau)$ and $Z_* = (\varepsilon R(r, \phi, \tau, \varepsilon),\ \omega + \phi(r, \phi, \tau, \varepsilon), 1)$. Since Z_* is a polynomial in ε, it has an expansion

$$Z_*(z, \varepsilon) = \sum_{i=0}^{K} \left(\frac{\varepsilon^i}{i!}\right) Z_i^0(z). \tag{3.6}$$

The change of variables $z = z(\zeta, \varepsilon)$ which performs the averaging process on (3.5) is constructed as the solution of a system of equations

$$\frac{dz}{d\varepsilon} = W(z, \varepsilon), \qquad z(0) = \zeta, \tag{3.7}$$

where W has a finite expansion

$$W(z, \varepsilon) = \sum_{i=0}^{N} \left(\frac{\varepsilon^i}{i!}\right) W_{i+1}(z). \tag{3.8}$$

The equation (3.5) in new coordinates becomes

$$\dot{\zeta} = Z^*(\zeta, \varepsilon) + O(\varepsilon^{N+1}), \tag{3.9}$$

where Z^* has an expansion of the form

$$Z^*(\zeta, \varepsilon) = \sum_{i=0}^{N} \left(\frac{\varepsilon^i}{i!}\right) Z_0^i(\zeta). \tag{3.10}$$

The method of Lie transforms introduces a double index array of functions $\{Z_j^i\}$ which agree with the previous definition when i or j are zero and can be computed by the recursive formulas

$$Z_j^i = Z_{j+1}^{i-1} + \sum_{k=0}^{j} \binom{j}{k} L_{k+1} Z_{j-k}^{i-1}, \tag{3.11}$$

where

$$L_{k+1} Z = \frac{\partial Z}{\partial z} W_{k+1} - \frac{\partial W_{k+1}}{\partial z} Z. \tag{3.12}$$

With this brief summary we shall prove:

THEOREM 2. *For any $N > 0$ there exists a change of variables $\rho = \bar{\rho}(r, \phi, t, \varepsilon)$, $\theta = \bar{\theta}(r, \phi, t, \varepsilon)$ with the properties*

(i) *$\bar{\rho}$ and $\bar{\theta}$ are 2π-periodic in ϕ and t and π-periodic in (θ, t);*

(ii) *$\bar{\rho}(r, \phi, t, 0) = r$ and $\bar{\theta}(r, \phi, t, 0) = \phi$;*

(iii) *when r, ϕ and ρ, θ are considered as polar coordinates in the plane, the change of variables is analytic in the rectangular coordinates of the plane, t and ε;*

(iv) *in the new coordinates the equations* (3.4) *become*

$$\begin{aligned} \dot{\rho} &= \varepsilon P(\rho, \theta, t, \varepsilon) + \varepsilon^{N+1} P'(\rho, \theta, t, \varepsilon), \\ \dot{\theta} &= \omega + \varepsilon H(\rho, \theta, t, \varepsilon) + \varepsilon^{N+1} H'(\rho, \theta, t, \varepsilon), \end{aligned} \tag{3.13}$$

where P, P', H, H' are 2π-periodic in θ and t and π-periodic in (θ, t) and

$$\begin{aligned} P(\rho, \theta + \omega t, t, \varepsilon) &= P(\rho, \theta, 0, \varepsilon), \\ H(\rho, \theta + \omega t, t, \varepsilon) &= H(\rho, \theta, 0, \varepsilon). \end{aligned} \tag{3.14}$$

Proof. We shall construct W inductively so that the conditions of Theorem 2 are satisfied order by order. Let $W_i = (u_i(r, \phi, \tau), v_i(r, \phi, \tau), 0)$ and $Z_j^i = (R_j^i(r, \phi, \tau), \phi_j^i(r, \phi, \tau), \delta)$, where $\delta = 1$ if $i = j = 0$ and $\delta = 0$ otherwise. Since we take the third component of W_i to be zero, then $t = \tau$. Note that $Z_0^0 = (0, \omega, 1)$ and Z_j^0 is 2π-periodic in θ and τ and π-periodic in (θ, τ).

Induction hypothesis M. Z_j^i and W_k are known for $0 \leqq i+j \leqq M$ and $0 \leqq k \leqq M$ and such that (a) Z_j^i and W_k are 2π-periodic in θ and τ and π-periodic in (θ, τ) (b) Z_j^i and W_k have finite Fourier expansions in (θ, τ) and (c) $Z_0^i = (R_0^i(r, \theta, \tau), \Phi_0^i(r, \theta, \tau), \delta)$, where R_0^i and Φ_0^i satisfy (3.14).

Clearly the induction hypothesis holds for $M = 0$, so assume it holds for $M = N - 1$. By (3.11),

$$Z_{N-1}^1 = \left\{ Z_N^0 + \sum_{k=0}^{N-2} \binom{N-1}{k} L_{k+1} Z_{N-k-1}^0 \right\} + L_n Z_0^0, \tag{3.15}$$

where the last term in the above has been separated out since it is the only term not given by the induction hypothesis or by the hypothesis of the theorem itself. First, the last component of Z_N^0 is zero. Z_n^0 is 2π-periodic in θ and τ, is π-periodic in (θ, τ) and has a finite Fourier expansion in θ and τ by the hypotheses of the theorem. Using (3.12) and the induction hypothesis $N-1$, it is clear that the other terms in the brackets have the same properties. Thus for $k=1$,

$$Z_{N-k}^{k}=K_{N-k}^{k}+L_{N-1}Z_0^0, \tag{3.16}$$

where K_{N-k}^k is a known function by the hypothesis with last component zero, 2π-periodic in θ and τ, π-periodic in (θ, τ) and has a finite Fourier series expansion. A simple induction argument shows the above is true for $k=1, 2, \cdots, N$. Thus we arrive at the basic equation to be solved, namely

$$Z_0^N=K_0^N+L_NZ_0^0. \tag{3.17}$$

If we let $K_0^N=(\alpha(r, \theta, \tau), \beta(r, \theta, \tau), 0)$, then (3.17) is equivalent to

$$\begin{aligned} R_0^N&=\alpha-\left[\omega\frac{\partial u_N}{\partial\phi}+\frac{\partial u_N}{\partial\tau}\right],\\ \Phi_0^N&=\beta-\left[\omega\frac{\partial v_N}{\partial\phi}+\frac{\partial v_N}{\partial\tau}\right]. \end{aligned} \tag{3.18}$$

The two equations in (3.18) are similar and so we shall only discuss the first.

Let

$$\begin{aligned} \alpha(r, \theta, \tau)&=a_{00}+\sum_S\{a_{pq}\cos(p\theta+q\tau)+b_{pq}\sin(p\theta+q\tau)\},\\ u_N(r, \theta, \tau)&=\sum_{S'}\{a'_{pq}\cos(p\theta+q\tau)+b'_{pq}\sin(p\theta+q\tau)\},\\ R_0^N(r, \theta, \tau)&=a''_{00}+\sum_{S''}\{a''_{pq}\cos(p\theta+q\tau)+b''_{pq}\sin(p\theta+q\tau)\}, \end{aligned} \tag{3.19}$$

where S is a finite set of pairs of integers $(p, q)\neq(0, 0)$. Since α is π-periodic in (θ, τ), we may assume that if $(p, q)\in S$, then $p+q$ is even. With the above expansions, the first equation in (3.18) becomes

$$\begin{aligned} a_{00}&=a''_{00},\\ a''_{pq}&=a_{pq}-(\omega p+q)b'_{pq},\\ b''_{pq}&=b_{pq}+(\omega p+q)a'_{pq}. \end{aligned} \tag{3.20}$$

The unprimed variables are given and the others must be solved. Define $S'=\{(p, q)\in S:(\omega p+q)\neq 0\}$ and $S''=S-S'$. Then define $b'_{pq}=(\omega p+q)^{-1}a_{pq}$, $a'_{pq}=-(\omega p+q)^{-1}b_{pq}$, $a''_{pq}=b''_{pq}=0$ when $(p, q)\in S'$. Also define $a''_{pq}=a_{pq}$, $b''_{pq}=b_{pq}$, $b'_{pq}=a'_{pq}=0$ when $(p, q)\in S''$. This clearly solves (3.18).

Since $(p, q)\in S$ implies $p+q$ even, and $S', S''\in S$ we have that $(p, q)\in S'$ or S'' implies $p+q$ even. Thus R_0^N and u_N are π-periodic in (θ, τ).

Also by the definition of S'', if $(p, q)\in S''$, then $p\omega+q=0$ and so $p(\theta+\omega\tau)+q\tau=p\theta$. Thus each term in the expansion for R_0^N satisfies (3.14).

4. Examples. We shall consider two classes of examples; the first is a modified form of Duffing's equation and the second is the forced van der Pol equation.

The modified Duffing equation. Consider the system

$$\ddot{x}+(\omega_0^2+\varepsilon^\alpha\Delta)x=\varepsilon\{\beta(x+A\cos t)^2+(x+A\cos t)^3\}. \tag{4.1}$$

When $\beta = 0$ we have the standard Duffing equation which admits the symmetry property previously discussed. When $\beta \neq 0$ the equation does not possess the symmetry property. Since we are interested in discussing bifurcations we shall choose ω_0 of the form $\omega_0 = p/q$, where p and q are relatively prime integers. We shall, in the series of examples given below, contrast and compare the types of bifurcations which occur when $\beta = 0$ and $\beta \neq 0$, when p and q are both odd and p or q even.

Note that when $\varepsilon = 0$, (4.1) has $x \equiv 0$ as a 2π-periodic solution. If $\beta \neq 0$ (i.e., we consider the equation as not possessing the special symmetry condition), then the characteristic multipliers of $x \equiv 0$ for $\varepsilon = 0$ are $\exp(\pm 2\pi\omega_0 i)$ which are not 1 unless $\omega_0 = k$, $k = 1, 2, 3, \cdots$. Thus for $\omega_0 \neq k$, $\beta \neq 0$ and ε sufficiently small, (4.1) has a 2π-periodic solution of order ε. We shall call this the harmonic solution. If $\beta = 0$ (i.e., we consider the equation as possessing the symmetry condition), then the characteristic multipliers of the odd harmonic solution $x = 0$ are $-\exp(\pm\pi\omega_0 i)$ which are not $+1$ unless $\omega_0 = 2k-1$, $k = 1, 2, \cdots$. Thus if $\omega_0 \neq 2k-1$, $k = 1, 2, 3, \cdots$, and ε is sufficiently small, (4.1) will have an odd harmonic 2π-periodic solution of order ε.

Note that the equation is Hamiltonian and so the product of the characteristic multipliers must be $+1$ for all periodic solutions. If the characteristic multipliers are real and distinct, the periodic solution will be called hyperbolic, and if the characteristic multipliers are of unit modulus and $\neq \pm 1$, the periodic solution will be called elliptic. Since the characteristic multipliers are continuous in a parameter, small perturbations of elliptic (or hyperbolic) periodic solutions will remain elliptic (resp. hyperbolic).

In the examples given below we shall discuss the bifurcations from the harmonic periodic solution for the truncated averaged equation only. It is only a simple application of the implicit function theorem to establish that these periodic solutions persist for the full equation. Most of the results are covered by the theorem in Henrard [5].

The examples given below are arranged in groups which compare and contrast bifurcation phenomenon. Examples 1 to 3 are one group.

Example 1. $\omega_0 = \frac{1}{2}$, $\alpha = 1$, $\beta \neq 0$. In this case the average of (4.1) is

$$\begin{aligned} \dot{r} &= -\varepsilon\{r\beta A \sin(2\phi - t)\} + O(\varepsilon^2), \\ \dot{\phi} &= \tfrac{1}{2} + \varepsilon\{\Delta - \tfrac{3}{2}A^2 - \tfrac{3}{4}r^2 - A\beta\cos(2\phi - t)\} + O(\varepsilon^2). \end{aligned} \tag{4.2}$$

As remarked above, (4.1) has a 2π-periodic solution, the harmonic, which is of order ε^1 in the original coordinates. From the equation for $\dot{r}$ in (4.2) one sees that in the new coordinates the harmonic is of order ε^2. Thus the change of coordinates which translates the harmonic to the origin in (4.2) differs from the identity by a term which is 2π-periodic and of order ε^2. Having made this change of variables, the terms of order 1 and ε^1 in (4.2) would be unchanged but the equation for $\dot{r}$ would contain r as a factor! Thus we may assume without loss of generality that the equation for $\dot{r}$ in (4.2) has r as a common factor. We note that this change of variables can and will be assumed to be made on all the examples of this section. This change of variables precisely locates the harmonic and so simplifies the discussion of the bifurcations.

When $\varepsilon=0$, the harmonic has characteristic multipliers $-1, -1$. In order to determine the characteristic multipliers of the harmonic for $\varepsilon \neq 0$, it is necessary to calculate the Jacobian K of the period map in rectangular coordinates at the origin. Equations (4.2) are easy to solve to first order, and then after changing to rectangular coordinates one finds that

$$K=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}+\varepsilon\begin{pmatrix}0 & (a-b)\\ (a+b) & 0\end{pmatrix}+\varepsilon^2\begin{pmatrix}m & n\\ r & s\end{pmatrix}+O(\varepsilon^3), \tag{4.3}$$

where $a=\beta A\pi$, $b=\{\frac{3}{2}A^2-\Delta\}\pi$ and m, n, r, s are constants which need not be calculated. Since $\det K\equiv 1$, one calculates that $m+s=b^2$ and so the trace is

$$K=-2+\varepsilon^2\{(\Delta-\tfrac{3}{2}A^2)^2-\beta^2A^2\}\pi^2+\cdots.$$

Consider A and β as fixed nonzero numbers and Δ as a parameter to be varied. By applying the implicit function theorem to the equation trace $K=-2$, there exist functions $\Delta_1(\varepsilon), \Delta_2(\varepsilon)$ such that $\Delta_1(0)=\frac{3}{2}A^2-|\beta A|$, $\Delta_2(\varepsilon)=\frac{3}{2}A^2+|\beta A|$ and $|\text{trace } K|>2$ when $\Delta\in(\Delta_1(\varepsilon), \Delta_2(\varepsilon))$ and trace $K=-2$ when $\Delta=\Delta_1(\varepsilon)$ or $\Delta_2(\varepsilon)$. Thus as we change the natural frequence, i.e., change Δ, the harmonic changes from elliptic to hyperbolic and back to elliptic.

In order to find 4π-periodic solutions which bifurcate from the harmonic, let $r(t, r_0, \phi_0, \varepsilon), \phi(t, r_0, \phi_0, \varepsilon)$ be the solution of (4.2) which pass through r_0, ϕ_0 when $t=0$. The bifurcation equations are

$$\begin{aligned}&(4\pi\varepsilon r_0)^{-1}\{r(4\pi, r_0, \phi_0, \varepsilon)-r_0\}=-\beta A\sin 2\phi_0\cdots=0,\\ &(4\pi\varepsilon)^{-1}\{\phi(4\pi, r_0, \phi_0, \varepsilon)-\phi_0-2\pi\}=\{\Delta-\tfrac{3}{2}A^2-\tfrac{3}{4}r_0^2-A\beta\cos 2\phi_0\}+\cdots\\ &=0.\end{aligned} \tag{4.4}$$

Note that we have factored out an r_0 from the first equation which is admissible since the harmonic is at the origin. It is easy to find solutions to (4.4) to first order and then apply the implicit function theorem to obtain solutions $(r_i(\varepsilon), \phi_i(\varepsilon))$, $i=1, 2, 3, 4$, of (4.4), where

$$\phi_1(0)=0, \quad \phi_2(0)=\pi, \quad \phi_3(0)=\pi/2, \quad \phi_4(0)=3\pi/2 \tag{4.5}$$

and

$$\begin{aligned}r_1(0)=r_2(0)&=\tfrac{4}{3}(\Delta-\tfrac{3}{2}A^2-A\beta),\\ r_3(0)=r_4(0)&=\tfrac{4}{3}(\Delta-\tfrac{3}{2}A^2+A\beta).\end{aligned} \tag{4.6}$$

These solutions tend to the harmonic as $\Delta\to\Delta_1(\varepsilon)$ or $\Delta\to\Delta_2(\varepsilon)$. One calculates as before that the trace of the Jacobian of the period map for these periodic solutions is $2+\varepsilon^2(3r_1(0)^2\beta A)+\cdots$ and $2+\varepsilon^2(-3r_2(0)^2\beta A)+\cdots$.

In summary: let A and β be fixed and nonzero. Let ε be sufficiently small. For $\Delta\notin[\Delta_1(\varepsilon), \Delta_2(\varepsilon)]$, the harmonic is elliptic and for $\Delta\in(\Delta_1(\varepsilon), \Delta_2(\varepsilon))$, the harmonic is hyperbolic. For $\Delta>\Delta_1(\varepsilon)$, there are two elliptic 4π-periodic solutions of (4.2) or (4.1) which tend to the harmonic as $\Delta\to\Delta_1(\varepsilon)+$, and for $\Delta>\Delta_2(\varepsilon)$ there are two hyperbolic 4π-periodic solutions of (4.2) or (4.1) which tend to the harmonic as $\Delta\to\Delta_2(\varepsilon)+$.

It is easy to see that the two elliptic (resp. hyperbolic) 4π-periodic solutions are really 2π-translates of one another and so should be considered as a single periodic solution.

This is the typical type of bifurcation which occurs in a Hamiltonian system at a periodic solution which has characteristic multipliers $-1, -1$ (cf. [9]).

The next example shows that a far different type of bifurcation occurs when $\omega_0 = \frac{1}{2}$, $\beta = 0$, but this can easily be explained by the general results of § 2.

Example 2. $\omega_0 = \frac{1}{2}$, $\alpha = 1$, $\beta = 0$. Since $\beta = 0$, equation (4.1) is symmetric and so the harmonic is odd harmonic. Using our convention, the characteristic multipliers of the harmonic when $\varepsilon = 0$ are $e^{\pm\pi i/2} = \pm i$, whereas the usual convention gives that the characteristic multipliers are $-1, -1$. In view of this, one would not expect the harmonic to become hyperbolic when $\varepsilon \neq 0$ as in the previous example. Since $\beta = 0$, one must compute more terms in the average of (4.1) to yield

$$\begin{aligned} \dot r &= \varepsilon^2\{-\tfrac{33}{8}r^3A^3 \sin(4\phi - 2t)\} + O(\varepsilon^3), \\ \dot\phi &= \tfrac{1}{2} + \varepsilon\{\Delta - \tfrac{3}{2}A^2 - \tfrac{3}{4}r^2\} + O(\varepsilon^2). \end{aligned} \tag{4.7}$$

As before we may assume that the harmonic is given by $r \equiv 0$ in (4.7). It is easy to compute the characteristic multipliers of the harmonic from (4.7) to find that they are

$$\exp(\pm\pi i)\{\tfrac{1}{2} + \varepsilon(\Delta - \tfrac{3}{2}A^2) + O(\varepsilon^2)\}. \tag{4.8}$$

Thus by the implicit function theorem, there is a function $\Delta_1(\varepsilon)$ such that $\Delta_1(0) = \frac{3}{2}A^2$, and the harmonic has characteristic multiplier $\pm i$ if and only if $\Delta = \Delta_1(\varepsilon)$.

Now we shall look for odd harmonic 4π-periodic solutions. The bifurcation equations are then

$$\begin{aligned} (2\pi\varepsilon^2 r_0^3)^{-1}(r(2\pi, r_0, \phi_0, \varepsilon) - r_0) &= -\tfrac{33}{8}A^3 \sin 4\phi_0 + \cdots = 0, \\ (2\pi\varepsilon)^{-1}(\phi(2\pi, r_0, \phi_0, \varepsilon) - \phi_0 - \pi) &= (\Delta - \tfrac{3}{2}A^2 - \tfrac{3}{4}r_0^2) + \cdots = 0. \end{aligned} \tag{4.9}$$

As before one solves these equations to first order and applies the implicit function theorem to get solutions $(r_i(\varepsilon), \phi_i(\varepsilon))$, $i = 1, \cdots, 8$, where

$$\begin{aligned} &\phi_1(0) = 0, \quad \phi_2(0) = \pi/2, \quad \phi_3(0) = \pi, \quad \phi_4(0) = 3\pi/2, \\ &\phi_5(0) = \pi/4, \quad \phi_6(0) = 3\pi/4, \quad \phi_7(0) = 5\pi/4, \quad \phi_8(0) = 7\pi/4, \end{aligned} \tag{4.10}$$

and

$$r_i(0) = \tfrac{4}{3}(\Delta - \tfrac{3}{2}A^2). \tag{4.11}$$

As before these solutions can be arranged into groups containing trajectories which are translates by 2π of one another. Specifically the solutions corresponding to the pairs of indices (1, 3), (2, 4), (5, 7) and (6, 8) forms such groups. But the change of variables discussed in § 2 makes the second (resp. the fourth) group the translate of the first (resp. the third) by π. We shall thus group together the first four and the last four solutions in order to stress the similarity of the results with the typical bifurcation of a periodic orbit whose characteristic multipliers are the eigenvalue of $-X(\pi)$ rather than of $X(2\pi)$. While doing so we shall keep in mind that each group contains two different periodic orbits of the original system.

This is the typical bifurcation which occurs in a Hamiltonian system when a periodic solution has characteristic multipliers which are fourth roots of unity [9]. Compare this bifurcation with the bifurcation discussed in the next example where $\beta \neq 0$, $\omega_0 = \frac{1}{4}$.

Example 3. $\omega_0 = \frac{1}{4}$, $\alpha = 1$, $\beta \neq 0$. The equation is not symmetric and the characteristic multipliers of the harmonic for $\varepsilon = 0$ are $\pm i$, fourth roots of unity. The averaged equations are

$$\begin{aligned} \dot{r} &= \varepsilon^2 20 r^3 A\beta \sin(4\phi - t) + O(\varepsilon^3), \\ \dot{\phi} &= \tfrac{1}{4} + \varepsilon\{2\Delta - 3A^2 - \tfrac{3}{2}r^2\} + O(\varepsilon^2). \end{aligned} \tag{4.12}$$

The analysis of this example is almost the same as in the previous example. In summary: there exists a $\Delta_1(\varepsilon)$ such that $\Delta_1(0) = \frac{3}{2}A^2$ and the harmonic has characteristic multipliers equal to $\pm i$ for $\Delta = \Delta_1(\varepsilon)$, ε small. For $\Delta > \Delta_1(\varepsilon)$, there are two groups (of four each) of 8π-periodic solutions for $\Delta > \Delta_1(\varepsilon)$: one group is elliptic and the other hyperbolic, which tend to the harmonic as $\Delta \to \Delta_1(\varepsilon)+$.

The next group of examples is concerned with the bifurcations from the harmonic solution when the characteristic multipliers are third roots of unity. The case when $\omega_0 = \frac{1}{3}$, $\beta = 0$, $\alpha = 1$ is analyzed in many standard texts, and so we shall assume that the reader is familiar with the example. See [2] or [3]. Recall that in this case two groups of three unstable periodic solutions bifurcate from the harmonics, one for Δ greater than some $\Delta_1(\varepsilon)$ and the other one for Δ smaller than $\Delta_1(\varepsilon)$.

Example 4. $\omega_0 = \frac{2}{3}$, $\alpha = 1$, $\beta \neq 0$. The equation is without symmetry and the harmonic has characteristic multipliers $e^{\pm 4\pi i/3}$, cube roots of unity, when $\varepsilon = 0$. The averaged equations are

$$\begin{aligned} \dot{r} &= -\varepsilon^2 (\tfrac{4455}{1024}) r^2 A^2 \beta \sin(3\phi - 2t) + O(\varepsilon^3), \\ \dot{\phi} &= \tfrac{2}{3} + \varepsilon\{\tfrac{3}{4}\Delta - \tfrac{9}{8}A^2 - \tfrac{9}{16}r^2\} + O(\varepsilon^2). \end{aligned} \tag{4.13}$$

The analysis is similar to the analysis of Example 2. In summary: there exists a $\Delta_1(\varepsilon)$, $\Delta_1(0) = \frac{3}{2}A^2$ such that the harmonic has characteristic multipliers $e^{\pm 4\pi i/3}$ when $\Delta = \Delta_1(\varepsilon)$ for ε small. For $\Delta > \Delta_1(\varepsilon)$, there are two groups (of three each) of periodic solution of period 6π: one group is elliptic and one hyperbolic which tend to the harmonic as $\Delta \to \Delta_1(\varepsilon)+$.

This is again the typical behavior when one considers bifurcation from a periodic solution which has characteristic multipliers which are cube roots of unity.

Example 5. $\omega_0 = \frac{2}{3}$, $\alpha = 1$, $\beta = 0$. When $\omega_0 = \frac{1}{3}$ the standard convention and our convention on characteristic multipliers both say that the harmonic has characteristic multipliers which are cube roots of unity for $\varepsilon = 0$. In the case when $\omega_0 = \frac{2}{3}$, the usual convention still gives that the characteristic multipliers of the harmonic are cube roots of unity but our convention gives $-\exp(\pm 2\pi i/3)$ which is a sixth root of unity.

In this case, the averaged equations are

$$\begin{aligned}\dot r &= \varepsilon^4(-43.8292)r^5A^4\sin(6\phi-4t)+O(\varepsilon^5),\\ \dot\phi &= \tfrac{2}{3}+\varepsilon\{\tfrac{3}{4}\Delta-\tfrac{9}{8}A^2-\tfrac{9}{16}r^2\}+O(\varepsilon^2).\end{aligned} \tag{4.14}$$

Using our convention again in summary: there exists a $\Delta_1(\varepsilon)$, $\Delta_1(0)=\frac{3}{2}A^2$ such that the harmonic has characteristic multipliers $-\exp(\pm 2\pi i/3)$ for $\Delta=\Delta_1(\varepsilon)$, ε small. For $\Delta>\Delta_1(\varepsilon)$, there are two groups (of 6 each) of 6π-periodic solutions: one group is elliptic and one hyperbolic, which tend to the harmonic as $\Delta\to\Delta_1(\varepsilon)+$. The grouping of solutions is made according to the rules explained in the analysis of Example 2.

In the next four examples we simply give the averaged equations for the readers reference. Enough terms in the averaged equations are given so that the reader can easily analyze the equations. Some coefficients are truncated to 6 significant digits even though the original computations were carried out to 16 significant digits.

Example 6. $\omega_0=3$, $\alpha=1$, $\beta=0$. The average equations are

$$\begin{aligned}\dot r &= -\varepsilon(\tfrac{25}{6})r^3\sin(\phi-3t)+O(\varepsilon^2),\\ \dot\phi &= 3+\varepsilon\{\tfrac{1}{6}\Delta-\tfrac{1}{4}A^2-\tfrac{1}{8}r^2-\tfrac{25}{6}r^{-1}A^3\cos(\phi-3t)\}+O(\varepsilon^2).\end{aligned}$$

To this order the average is the same if $\beta\neq 0$.

Example 7. $\omega_0=2$, $\alpha=1$, $\beta\neq 0$. The averaged equations are

$$\begin{aligned}\dot r &= -\varepsilon(\tfrac{1}{8})r^2\beta\sin(\phi-2t)+O(\varepsilon^2),\\ \dot\phi &= 2+\varepsilon\{\tfrac{1}{4}\Delta-\tfrac{3}{8}A^2-\tfrac{3}{64}r^2-\tfrac{1}{8}r^{-1}A^2\beta\cos(\phi-2t)\}+O(\varepsilon^2).\end{aligned}$$

Example 8. $\omega_0=2$, $\alpha=1$, $\beta=0$. The averaged equations are

$$\begin{aligned}\dot r &= -\varepsilon^2(\tfrac{21}{1280})rA^4\sin(2\phi-4t)+O(\varepsilon^3),\\ \dot\phi &= 2+\varepsilon\{\tfrac{1}{4}\Delta-\tfrac{3}{8}A^2-\tfrac{3}{64}r^2\}+O(\varepsilon^2).\end{aligned}$$

Example 9. $\omega_0=4$, $\alpha=1$, $\beta=0$. The averaged equations are

$$\begin{aligned}\dot r &= -\varepsilon^4(3.74923\times 10^{-5})rA^8\sin(2\phi-8t)+O(\varepsilon^5),\\ \dot\phi &= 4+\varepsilon\{\tfrac{1}{8}\Delta-\tfrac{3}{16}A^2-\tfrac{3}{32}r^2\}+O(\varepsilon^2).\end{aligned}$$

Locking-in and van der Pol's equation. Consider the equation

$$\ddot x+(\omega_0^2+\varepsilon^\alpha\Delta)x=\varepsilon\{(1-x^2)\dot x+A\cos t\}. \tag{4.15}$$

The theory of invariant manifolds can be applied to (4.15) when $\omega_0\neq 1$ to establish the existence of an invariant periodic cylinder in R^3—the $(x,\dot x,t)$—space (cf. Hale [2], [3]). Loud [8] has investigated this equation when $\omega_0=1$, $\alpha=1$ and established the existence of periodic solutions of period 2π for $\Delta\neq 0$. Thus equation (4.15) has periodic solutions whose frequencies are "locked-in" to the period of the forcing term even though the natural frequence, $1+\varepsilon\Delta$, is not precisely one. In view of the general discussion of the locking-in phenomenon in Sternberg [10], one would expect something similar when ω_0 is a rational number. Indeed this is the case as the examples given below show.

Example 10. $\omega_0 = \frac{1}{3}$, $\alpha = 2$. The averaged equations are

$$\dot{r} = \varepsilon(\tfrac{1}{2})\{r - \tfrac{1}{4}r^3\} + \varepsilon^2\{\tfrac{9}{64}r^2 A \cos(3\phi - t)\} + O(\varepsilon^3),$$
$$\dot{\phi} = \tfrac{1}{3} + \varepsilon^2\{\tfrac{3}{2}\Delta - \tfrac{3}{8} + \tfrac{9}{16}r^2 - \tfrac{33}{256}r^4 - \tfrac{9}{64}rA \sin(3\phi - t)\} + O(\varepsilon^3).$$

Neglecting terms of order ε^2 and higher, these equations are autonomous and have a stable limit cycle with characteristic multipliers 1 and $1 - 3\pi\varepsilon + O(\varepsilon^2)$. This information is enough to apply the invariant manifold theory to give the existence of a periodic invariant cylinder but not enough to establish the existence of periodic or ergodic solutions on this invariant cylinder. However, the terms of order ε^2 do establish the existence of periodic solutions of period 6π as we shall see. From the above equations we can easily compute the approximate solutions to find that the bifurcation equations are

$$\varepsilon^{-1}\{r(6\pi, r_0, \phi_0) - r_0\} = 3\pi\{r_0 - \tfrac{1}{4}r^3\} + O(\varepsilon) = 0,$$
$$\varepsilon^{-2}\{\phi(6\pi, r_0, \phi_0) - \phi_0 - 2\pi\} = (6\pi)\{\tfrac{3}{2}\Delta - \tfrac{3}{8} + \tfrac{9}{16}r^2 - \tfrac{33}{256}r^4 - \tfrac{9}{64}rA \sin 3\phi_0\} + O(\varepsilon) = 0.$$

The first equation has a solution $r_0 = 2 + O(\varepsilon)$ by the implicit function theorem. Substituting this into the second equation gives

$$f(\Delta, \phi_0) + O(\varepsilon) = 0,$$

where $f(\Delta, \phi_0) = \Delta - \frac{1}{8} - \frac{3}{16}A \sin 3\phi_0$. The zeros of f are easy to find by solving for Δ as a function of ϕ_0. Since $(\partial f/\partial\Delta) = 1$, the implicit function theorem can be applied again to yield 6π-periodic solutions of van der Pol's equation. In summary one finds: for ε sufficiently small, there exist functions $\Delta_1(\varepsilon)$, $\Delta_2(\varepsilon)$, $\Delta_1(0) = \frac{1}{8} - \frac{3}{16}|A|$, $\Delta_2(0) = \frac{1}{8} + \frac{3}{16}|A|$ such that for $\Delta_1(\varepsilon) < \Delta < \Delta_1(\varepsilon)$, equation (4.15) has 6 periodic solutions of period 6π. These solutions are in two groups, 3 asymptotically stable and 3 unstable. The solutions in a group are translates of one another. Also for $\Delta = \Delta_1(\varepsilon)$ or $\Delta_2(\varepsilon)$, equation (4.15) has 3 periodic solutions of period 6π which has one characteristic multiplier equal to $+1$. For $\Delta < \Delta_1(\varepsilon)$ or $\Delta > \Delta_2(\varepsilon)$, the equation has no periodic solutions of least period 6π.

This gives another example of the locking-in phenomenon. The examples below are similar and so we shall only give the averaged equations.

Example 11. $\omega_0 = \frac{1}{2}$, $\alpha = 4$. The averaged equations are

$$\dot{r} = \varepsilon\{\tfrac{1}{2}r - \tfrac{1}{8}r^3\} + O(\varepsilon^3),$$
$$\dot{\phi} = \tfrac{1}{2} + \varepsilon^2\{-\tfrac{1}{4} + \tfrac{3}{8}r^2 - \tfrac{11}{128}r^4\} + \varepsilon^4\{\Delta - \tfrac{1}{16} + \tfrac{4}{9}A^2 + \tfrac{11}{32}r^2 - \tfrac{53}{135}r^2A^2$$
$$- \tfrac{763}{1536}r^4 + \tfrac{629}{3072}r^6 - (0.02570)r^8 - \tfrac{1}{36}r^2A^2 \cos(4\phi - 2t)\} + O(\varepsilon^5).$$

Example 12. $\omega_0 = \frac{2}{3}$, $\alpha = 4$. The averaged equations are

$$\dot{r} = \varepsilon\tfrac{1}{2}r - \tfrac{1}{8}r^3\} + O(\varepsilon^3),$$
$$\dot{\phi} = \tfrac{2}{3} + \cdots + \varepsilon^8(-7.456 \times 10^{-3})r^4A^4 \cos(6\phi - 4t) + O(\varepsilon^9).$$

In the above the terms indicated by three dots are polynomials in ε, Δ, r, A and independent of f and ϕ.

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