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Bounding Solutions of a Forced Oscillator

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To the memory of George Sell, our colleague and friend

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Abstract We give an alternate proof of a theorem in Wang and You (Z Angew Math Phys 47: 943–952, 1996) which shows that all solutions are bounded for a periodically forced nonlinear oscillator. Our proof relies on constructing an analytic change of variables by a convergent Lie series transformation to simplify the system so that the period map has large invariant curves by Moser's theorem.

Keywords Nonlinear oscillator · Bounded solutions · Invariant curve · Twist map

Mathematics Subject Classification 34C15 · 34K12 · 37E40 · 70H08

1 Introduction

In 1976 G. R. Morris [8] showed that all solutions of

$$\ddot{x} + 2x^3 = p(t),$$

are bounded when $p(t)$ is piecewise continuous and periodic. This gave rise to an abundance of generalizations [3–7, 12, 14] with references to many more. All these proofs depend on showing that Moser's invariant curve theorem [11] implies the existence of invariant curves near infinity for the period map.

Here we give an alternate proof of the generalization of Morris' result found in Wang and You [12]:

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Theorem 1 *For any integer $n > 1$ all solutions of*

$$\ddot{x} + nx^{2n-1} = p_0(t) + p_1(t)x + \cdots + p_{2n-2}(t)x^{2n-2} \tag{1}$$

are bounded where the coefficients of the polynomial on the right hand side are T-periodic and satisfy $p_0(t) \in C^0$, $p_j(t) \in C^1$ for $j = 1, \ldots, n-1$ and $p_j(t) \in C^2$ for $j = n, \ldots, 2n-2$.

Our proof like theirs is based on a generalization of action-angle variables and Moser's invariant curve theorem, but ours uses a convergent Lie transformation to simplify the general system. To begin we give a quick proof of Morris' original theorem. Then the full theorem is established in two steps. In the first step we define a special case and following the proof given for Morris' Theorem we prove boundedness for this case. In the second step we use the method of Lie transforms to construct an analytic change of variables to reduce the general case to the special case.

The method of Lie transforms was developed to simplify a system of differential equations by constructing a formal series in a small parameter ε by means of a generating function. Usually there is little hope of establishing convergence. However, in our case only a finite number of terms need to be simplified, so the generating function is a finite series in ε. Thus the change of variables which is the solution of a differential equation defined by the generating function is convergent for small ε. Finally by some straight forward estimates we show that the transformation converges all the way up to $\varepsilon = 1$.

The method of Lie transforms is an improvement over Birkhoff transformations, which have been used in the past to simplify Hamiltonian systems. Since [3] and with it [12] use Birkhoff transformations the contribution of our paper is to show how to prove their theorem with the help of a modern tool found in [2].

2 Action–Angle Variables

Let $\mathrm{sl}(\kappa)$ be the solution of the reference equation

$$\xi'' + n\xi^{2n-1} = 0, \quad \xi(0) = 0, \quad \xi'(0) = 1,$$

where $' = \dfrac{d}{d\kappa}$ and let $\mathrm{cl}(\kappa) = \mathrm{sl}'(\kappa)$. When $n = 1$ these are the standard sine and cosine functions and when $n = 2$ these are the lemniscate functions [13].

The Hamiltonian for this equation is

$$L = \frac{1}{2}\eta^2 + \frac{1}{2}\xi^{2n},$$

where $\eta = \xi'$, so $\mathrm{cl}^2(\kappa) + \mathrm{sl}^{2n}(\kappa) = 1$. Since the level sets of L are ovals these solutions are periodic. As κ increases from zero, $\mathrm{sl}(\kappa)$ increases from zero until it reaches its maximum value of 1 after some time $\tau > 0$ where

$$\tau = \int_0^1 \frac{d\xi}{\sqrt{1 - \xi^{2n}}}.$$

From the symmetry of the problem one sees that sl is an odd function which is even about τ so has the same basic symmetry as the sin function and cl is an even function which is odd about τ. Both functions are 4τ-periodic.

To get *action–angle* variables (K, κ), let

$$x = K^{\frac{1}{n+1}}\mathrm{sl}(\kappa), \qquad y = -K^{\frac{n}{n+1}}\mathrm{cl}(\kappa),$$

and check $dx \wedge dy = \frac{n}{n+1} dK \wedge d\kappa$, which is symplectic with multiplier $(n+1)/n$.

The Hamiltonian for Eq. (1) is

$$H = \frac{1}{2}(y^2 + x^{2n}) - \sum_{j=0}^{2n-2} p_j(t)\frac{x^{j+1}}{(j+1)}, \tag{2}$$

and in action-angle variables is

$$H = \frac{n+1}{2n} K^{\frac{2n}{n+1}} + \sum_{j=1}^{2n-1} K^{\frac{j}{n+1}} f_j(\kappa, t), \tag{3}$$

where we have set $f_j(\kappa, t) = -\dfrac{n+1}{jn}\ \mathrm{sl}^j(\kappa)\ p_{j-1}(t)$.

3 Proof of Morris' Theorem

In order to illustrate how Moser's invariant curve theorem can be applied with the action-angle coordinates (K, κ) in a simple setting let us prove Morris' original theorem. So look at the equation,

$$\ddot{x} + 2x^3 = p(t),$$

were $p(t)$ is piecewise continuous and T-periodic. In action-angle variables, (K, κ), the Hamiltonian is

$$H = \frac{3}{4}K^{4/3} - \frac{3}{2}K^{1/3}\mathrm{sl}(\kappa)p(t),$$

and the equations of motion are

$$\dot{K} = -\frac{3}{2}K^{1/3}\mathrm{cl}(\kappa)p(t),$$

$$\dot{\kappa} = -K^{1/3} + \frac{1}{2}K^{-2/3}\mathrm{sl}(\kappa)p(t).$$

Let $\Lambda = K^{1/3}$ so the equations become

$$\dot{\Lambda} = -\frac{1}{2}\Lambda^{-1}\mathrm{cl}(\kappa)p(t),$$

$$\dot{\kappa} = -\Lambda + \frac{1}{2}\Lambda^{-2}\mathrm{sl}(\kappa)p(t).$$

Note that since sl, cl, and p are all uniformly bounded these equations are analytic for all κ, t and $\Lambda > 0$.

Integrate from 0 to $-T$ to compute the period map $\mathcal{P} : (\Lambda, \kappa) \to (\Lambda^*, \kappa^*)$ were

$$\Lambda^* = \Lambda + F(\Lambda, \kappa),$$

$$\kappa^* = \kappa + T\Lambda + G(\Lambda, \kappa),$$

where $F(\Lambda, \kappa) = O(\Lambda^{-1}),\ G(\Lambda, \kappa) = O(\Lambda^{-2})$.

An *encircling curve* is a curve, $\mathcal{C}$, of the form $K = \phi(\kappa)$ (or $\Lambda = \phi(\kappa)$) were ϕ is continuous, 4τ-periodic, positive and near a circle about the origin. The invariant curve theorem requires the the image of an encircling curve must intersect itself. The coordinates (K, κ) are symplectic, so the period map is area preserving and thus the image under $\mathcal{P}$ of any encircling curve must intersect itself in the (K, κ) coordinates. The map from K to L is invertible from $K > 0$ to $L > 0$ so the image of an encircling curve in (Λ, κ) coordinates must intersect itself. The curve in invariant if $\mathcal{P}(\mathcal{C}) = \mathcal{C}$.

The invariant curve theorem requires that the functions F, G have at least ℓ derivatives were originally $\ell = 333$, but resent work has reduced it to $\ell = 5$. No matter our system is analytic and we only need a continuous invariant curve.

Let $\mathcal{A}(a)$ be the annulus $\{(\Lambda, \kappa) : 1 < a \le \Lambda \le a + 1\}$ define the ℓ^{th} norm of a function R on $\mathcal{A}(a)$ to be

$$|R|_\ell = \sup \left| \left(\frac{\partial}{\partial \Lambda}\right)^{\sigma_1} \left(\frac{\partial}{\partial \kappa}\right)^{\sigma_2} R(\Lambda, \kappa) \right|$$

where the sup is over all $0 \le \sigma_1 + \sigma_2 \le \ell$, and all $(\Lambda, \kappa) \in \mathcal{A}(a)$. From the form of the equations

$$|F|_\ell = O(a^{-1}), \quad |G|_\ell = O(a^{-2}).$$

Moser's theorem says there is a $\delta > 0$ depending on the given data such that if $|F|_\ell < \delta,\ |G|_\ell < \delta$ then there is an invariant encircling curve in $\mathcal{A}(a)$. From the above there is an a^* such that for all $a > a^*$ we have $|F|_\ell < \delta,\ |G|_\ell < \delta$ on $\mathcal{A}(a)$. So the period map $\mathcal{P}$ has arbitrarily large invariant encircling curves, so all solutions are bounded and thus Morris' Theorem in established.

4 The Special Case

Our special case is the Hamiltonian in action-angle variables of the form

$$\mathcal{H} = \frac{n+1}{2n} K^{\frac{2n}{n+1}} + \sum_{j=2}^{2n-1} K^{\frac{j}{n+1}} \overline{f}_j(t) + \sum_{j=-\infty}^{1} K^{\frac{j}{n+1}} f_j(\kappa, t), \tag{4}$$

where

(1) $\overline{f}_j(t)$ is continuous and T-periodic in t for $j = 2, \dots, 2n-1$,
(2) $f_j(\kappa, t)$ is analytic and 4τ-periodic in κ, continuous and T-periodic in t for $j = -\infty, \dots, 1$,
(3) the infinite series in (4) is uniformly convergent for $K \ge \mathbf{K}$ and all κ, t with $\mathbf{K} > 0$ a constant.

In $\mathcal{H}$ the κ dependence has been removed from some terms to facilitate the proof at a cost of many extra terms. The extra terms are created when the original Hamiltonian (3) is normalized in the next section.

Proposition 1 *All solutions of the equations with Hamiltonian (4) are bounded.*

Proof The equations of motion are

$$\dot{K} = \sum_{j=-\infty}^{1} K^{\frac{j}{n+1}} \frac{\partial f_j(\kappa, t)}{\partial \kappa},$$

$$\dot{\kappa} = -K^{\frac{n-1}{n+1}} - \sum_{j=2}^{2n-1} \left(\frac{j}{n+1}\right) K^{\frac{j-n-1}{n+1}} \overline{f}_j(t) - \sum_{j=-\infty}^{1} \left(\frac{j}{n+1}\right) K^{\frac{j-n-1}{n+1}} f_j(\kappa, t).$$

Set $\Lambda = K^{\frac{n-1}{n+1}}$ so that the differential equations are now

$$\dot{\Lambda} = \frac{n-1}{n+1} \sum_{j=-\infty}^{1} \Lambda^{\frac{j-2}{n-1}} \frac{\partial f_j(\kappa, t)}{\partial \kappa},$$

$$\dot{\kappa} = -\Lambda - \sum_{j=2}^{2n-1} \left(\frac{j}{n+1}\right) \Lambda^{\frac{j-n-1}{n-1}} \overline{f}_j(t) - \sum_{j=-\infty}^{1} \left(\frac{j}{n+1}\right) \Lambda^{\frac{j-n-1}{n-1}} f_j(\kappa, t).$$

The above are series in $\Lambda^{\frac{1}{n-1}}$ which are convergent for large Λ. The two infinite series will be treated as perturbations and the finite series in $\dot{\kappa}$ contributes to the twist term. The dominate term in the infinite series for $\dot{\Lambda}$ is of order $\Lambda^{\frac{-1}{n-1}}$ and for $\dot{\kappa}$ is of order $\Lambda^{\frac{-n}{n-1}}$.

We are interested in large K, that is large Λ so that

$$\dot{\Lambda} = O(\Lambda^{\frac{-1}{n-1}}),$$

$$\dot{\kappa} = -\Lambda - \sum_{j=2}^{2n-1} \left(\frac{j}{n+1}\right) \Lambda^{\frac{j-n-1}{n-1}} \overline{f}_j(t) + O(\Lambda^{\frac{-n}{n-1}}),$$

where the estimates are on $K \geq \mathbf{K}$. Integrating from 0 to $-T$ to compute the period map $\mathcal{P} : (\Lambda, \kappa) \to (\Lambda^*, \kappa^*)$ to be

$$\Lambda^* = \Lambda + F(\Lambda, \kappa),$$

$$\kappa^* = \kappa + \alpha(\Lambda) + G(\Lambda, \kappa),$$

where $F(\Lambda, \kappa) = O(\Lambda^{\frac{-1}{n-1}}),\ G(\Lambda, \kappa) = O(\Lambda^{\frac{-n}{n-1}})$. The twist term is

$$\alpha(\Lambda) = T\Lambda + \sum_{j=2}^{2n-1} \sigma_j \Lambda^{\frac{j-n-1}{n-1}}, \quad \text{with} \quad \sigma_j = \frac{j}{n+1} \int_0^{-T} \overline{f}_j(t) dt,$$

and its derivative is

$$\frac{d\alpha(\Lambda)}{d\Lambda} = T + \sum_{j=2}^{2n-1} \frac{j-n-1}{n-1} \sigma_j \Lambda^{\frac{j-2n}{n-1}} = T + O(\Lambda^{\frac{-1}{n-1}}).$$

Consider the annulus $\mathcal{A}(a) = \{(\Lambda, \kappa) : 1 < a \leq \Lambda \leq a + 1\}$. There is an $a^* > 1$ such that $\frac{1}{2}T < d\alpha(\Lambda)/d\Lambda < 2T$ on $\mathcal{A}(a)$ when $a > a^*$. Moser's theorem says there is a $\delta > 0$ depending on the given data such that if $|F|_\ell < \delta,\ |G|_\ell < \delta$ then there is an invariant encircling curve in $\mathcal{A}(a)$. From the above there is an $a^{**} > a^*$ such that for all $a > a^{**}$

we have $|F|_\ell < \delta,\ |G|_\ell < \delta$ on $\mathcal{A}(a)$. So the period map $\mathcal{P}$ has arbitrarily large invariant encircling curves, all solutions are bounded and the Proposition in established. □

5 Reduction to the Special Case

In this final section we will finish the proof of the Theorem 1 by proving

Proposition 2 *There exists an invertible symplectic change of variables which transforms the original Hamiltonian H in (3) to the special Hamiltonian $\mathcal{H}$ in (4).*

The change of variables will be a composition of two, the first removes the κ dependence in the terms where $n+1 \le j \le 2n-1$ and the second removes the κ dependence from the terms where $2 \le j \le n$. More precisely the first transforms the original Hamiltonian (3) into an intermediate Hamiltonian of the form

$$\mathcal{H}_1 = \frac{n+1}{2n} K^{\frac{2n}{n+1}} + \sum_{j=n+1}^{2n-1} K^{\frac{j}{n+1}} \overline{f}_j(t) + \sum_{j=-\infty}^{n} K^{\frac{j}{n+1}} f_j(\kappa, t). \tag{5}$$

The second transformation will transform intermediate Hamiltonian (5) into the special Hamiltonian $\mathcal{H}$ given in (4). As we shall see this two step approach forces the different differentiability requirements on the $p_j(t)$.

Note that $\overline{f}_j(t)$ and $f_j(\kappa, t)$ are generic functions which at each stage are periodic function with the same properties as described in the definition of $\mathcal{H}$. It should be noted that in $\mathcal{H}_1$ terms with $K^{\frac{j}{n+1}}$ still depend on κ when $j \le n$ whereas that statement holds in $\mathcal{H}$ for $j \le 1$.

Special care will be taken to show that each transformation is convergent, taking domain to domain and that the precise differentiability of the $p_i(t)$'s is observed. The change of variables is constructed by the method of Lie transforms of Deprit [2]. See [9,10] for the complete details of the Lie transformation method and for the source for our notation.

To this end we introduce a parameter ε and consider the Hamiltonian

$$\begin{aligned} H_*(K, \kappa, t, \varepsilon) &= \frac{n+1}{2n} K^{\frac{2n}{n+1}} + \sum_{j=1}^{2n-1} K^{\frac{j}{n+1}} \varepsilon^{2n-j} f_j(\kappa, t), \\ &= \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!} H_i^0(K, \kappa, t), \end{aligned} \tag{6}$$

where

$$\begin{aligned} H_0^0 &= \tfrac{n+1}{2n} K^{\frac{2n}{n+1}}, \\ H_i^0 &= i! K^{\frac{2n-i}{n+1}} f_{2n-i}(\kappa, t), \qquad \text{for } i = 1, 2, \ldots, 2n-1, \\ H_i^0 &= 0, \qquad \text{for } i = 2n, 2n+1, \ldots. \end{aligned} \tag{7}$$

The parameter ε is usually consider small so that it generates a near identity transformation, but in our case the original Hamiltonian H in (3) is obtained from H_* in (6) by setting $\varepsilon = 1$. Therefore we need to construct the change of variables, which is valid and convergent when $\varepsilon = 1$. This is accomplished by taking only a finite number terms in the generating function W and with careful estimates.

The general procedure is to expand everything in the parameter ε and use the following notation. Introduce a double indexed array of functions H_j^i so that the Hamiltonian is H_* in (6) is transformed to the Hamiltonian

$$H^*(K,\kappa,t,\varepsilon)=\sum_{i=0}^{\infty}\frac{\varepsilon^i}{i!}H_0^i(K,\kappa,t). \tag{8}$$

The generating function for the transformation is

$$W(K,\kappa,t,\varepsilon)=\sum_{k=0}^{\infty}\frac{\varepsilon^k}{k!}W_{k+1}(K,\kappa,t). \tag{9}$$

One computes the transformation via a Lie triangle, whose entries are given by

$$H_j^i=H_{j+1}^{i-1}+\sum_{k=0}^{j}\binom{j}{k}\left\{H_{j-k}^{i-1},W_{k+1}\right\}. \tag{10}$$

The interdependence of the functions $\{H_j^i\}$ can easily be understood by considering the Lie triangle

$$\begin{array}{ccccc} H_0^0 & & & & \\ \downarrow & & & & \\ H_1^0 & \to & H_0^1 & & \\ \downarrow & & \downarrow & & \\ H_2^0 & \to & H_1^1 & \to & H_0^2 \\ \downarrow & & \downarrow & & \downarrow \end{array}$$

The coefficients of the expansion of the old function H_* are in the left column, and those of the new function H^* are on the diagonal. Formula (10) states that to calculate any element in the Lie triangle, one needs the entries in the column one step to the left and up.

Since the Hamiltonians depend on t the remainder function must be computed by a similar Lie triangle, which gives rise to differentiability requirements on the $p_j(t)$'s. In our case the dependency on t is not initially important and our goal is only to make the first few terms H_0^i independent of κ. As a benefit of this approach we can compute the remainder function R as the transformation of $-\partial W/\partial t$ after $W(K,\kappa,t,\varepsilon)$ has been determined.

Each transformation will be done in three steps. First the Lie transformation will be applied ignoring the t dependence, then the transformation is shown to be convergent up to $\varepsilon=1$, and finally the remainder term will be computed. The three steps given below are for the first transformation and then the modifications for the second transformation will be discussed.

5.1 The First Transformation

We will use the algorithm summarized in Theorem 10.3.1 of [10] and to that end we introduce three sequences of linear spaces $\mathcal{P}_r$, $\mathcal{Q}_r$, $\mathcal{R}_r$ where r is a row index, $r=0,1,\ldots$. Specifically

$\mathcal{P}_r$ is the set of all functions of the form $K^{\frac{2n-r}{n+1}}F(\kappa,t)$, *(Row terms),*

$\mathcal{Q}_r$ is the set of all functions of the form $K^{\frac{2n-r}{n+1}}\overline{F}(t)$, *(Reduced terms),*

$\mathcal{R}_r$ is the set of all functions of the form $K^{\frac{n-r+1}{n+1}}\tilde{F}(\kappa,t)$, *(W terms),*

where $F(\kappa,t)$ is 4τ-periodic in κ and T-periodic in t, $\tilde{F}(\kappa,t)$ is 4τ-periodic in κ with mean value zero and T-periodic in t, and $\overline{F}(t)$ is T-periodic in t.

Now check the hypotheses. Clearly $H_r^0\in\mathcal{P}_r$ and $\mathcal{Q}_r\subset\mathcal{P}_r$. To check that $\{\mathcal{P}_r,\mathcal{R}_s\}\subset\mathcal{P}_{r+s}$ let $A=K^{(2n-r)/(n+1)}F_r(\kappa,t)\in\mathcal{P}_r$ and $B=K^{(n-s+1)/(n+1)}\tilde{F}_s(\kappa,t)\in\mathcal{R}_s$. Since the functions F_r and F_s are generic functions it is enough to check the powers of K in

$$\{A,B\}=\frac{\partial A}{\partial K}\frac{\partial B}{\partial\kappa}-\frac{\partial A}{\partial\kappa}\frac{\partial B}{\partial K}.$$

The powers are

$$\left(\frac{2n-r}{n+1}-1\right)+\left(\frac{n-s+1}{n+1}\right)=\left(\frac{2n-r}{n+1}\right)+\left(\frac{n-s+1}{n+1}-1\right)=\frac{2n-r-s}{n+1}$$

and therefore $\{A, B\} \in \mathcal{P}_{r+s}$.

Next we need to show that for any $D \in \mathcal{P}_r$ there is a solution pair $B \in \mathcal{Q}_r,\ C \in \mathcal{R}_r$ that satisfy the Lie equation

$$B = D + \{H_0^0, C\}.$$

Given $D = K^{\frac{2n-r}{n+1}} F_r(\kappa, t)$ define $B = K^{\frac{2n-r}{n+1}} \overline{F}_r(t)$ where $\overline{F}_r(t)$ is the κ mean value of $F_r(\kappa, t)$ and seek $C = K^{\frac{n-r+1}{n+1}} \tilde{F}_r(\kappa, t)$. We need to solve

$$0 = K^{\frac{2n-r}{n+1}}(F_r(\kappa, t) - \overline{F}_r(t)) + K^{\frac{2n-r}{n+1}} \frac{\partial \tilde{F}_r}{\partial \kappa}(\kappa, t),$$

and

$$\tilde{F}_r(\kappa, t) = -\int_0^{\kappa} (F_r(k, t) - \overline{F}_r(t))dk$$

does the trick and with it we have $W_r(K, \kappa, t) = K^{\frac{n-r+1}{n+1}} \tilde{F}_r(\kappa, t)$.

We stop computing new W terms after n rows and set $W_j = 0$ for $j \geq n$, so we have constructed a generating function

$$W(K, \kappa, t, \varepsilon) = \sum_{j=0}^{n-1} \frac{\varepsilon^j}{j!} W_{j+1}(K, \kappa, t) = \sum_{j=0}^{n-1} \frac{\varepsilon^j}{j!} K^{\frac{n-j}{n+1}} \tilde{F}_j(\kappa, t).$$

Thus W transforms (6) to (8) where the terms are of the form

$$H_0^0 = \frac{n+1}{2n} K^{\frac{2n}{n+1}}, \tag{11}$$

$$H_0^j = K^{\frac{2n-j}{n+1}} \overline{F}_j(t) \qquad \text{for } j = 1, \ldots, n, \tag{12}$$

$$H_0^j = K^{\frac{2n-j}{n+1}} F_j(\kappa, t) \qquad \text{for } j = n+1, \ldots, \infty. \tag{13}$$

So far the Lie procedure is formal, but the constructed generating function W is finite so is a convergent series. Moreover closer inspection reveals that the following Lemma applies.

Lemma 1 *There exists a constant* $\mathbf{K} > 0$ *such that the transformation generated by* $W(K, \kappa, t, \varepsilon)$ *is uniformly convergent for* $K > \mathbf{K}$, $0 \leq \varepsilon \leq 1$, *all* κ *and* t. *In particular the transformation takes* H_* *to* H^* *when* $\varepsilon = 1$.

Proof Look at the K equation for the transformation

$$\frac{dK}{d\varepsilon} = \frac{\partial W}{\partial \kappa} = \sum_{j=0}^{n-1} \frac{\varepsilon^j}{j!} K^{\frac{n-j}{n+1}} \frac{\partial \tilde{F}_j}{\partial \kappa}(\kappa, t),$$

$$(n+1)\frac{dK^{\frac{1}{n+1}}}{d\varepsilon} = \sum_{j=0}^{n-1} \frac{\varepsilon^j}{j} K^{\frac{-j}{n+1}} \frac{\partial \tilde{F}_j}{\partial \kappa}(\kappa, t),$$

or with $\Lambda = K^{\frac{1}{n+1}}$

$$(n+1)\frac{d\Lambda}{d\varepsilon} = \sum_{j=0}^{n-1} \frac{\varepsilon^j}{j!} \Lambda^{-j} \frac{\partial \tilde{F}_j}{\partial \kappa}(\kappa, t).$$

Now let $(n+1)A = \max |\partial \tilde{F}_j / \partial \kappa|$ so that

$$-A \sum_{j=0}^{n-1} \frac{\varepsilon^j}{j!} \Lambda^{-j} \leq \frac{d\Lambda}{d\varepsilon} \leq A \sum_{j=0}^{n-1} \frac{\varepsilon^j}{j!} \Lambda^{-j}.$$

Assume $K \geq 1$ and for $0 \leq \varepsilon \leq 1$ use $\sum_{j=0}^{n-1} \varepsilon^j / j! < B$ so that

$$-AB\Lambda^{1-n} \leq \frac{d\Lambda}{d\varepsilon} \leq AB.$$

Integrating these inequalities gives

$$\Lambda_0^n AB\varepsilon \leq \Lambda^n \leq (\Lambda_0 + AB\varepsilon)^n.$$

Thus if $\Lambda_0^n \geq 1 + nAB$ or $K_0 \geq (1 + nAB)^{\frac{n+1}{n}}$ the equations can be integrated all the way up to $\varepsilon = 1$ and $K(\varepsilon) > 1$.

The κ equation for the transformation is

$$\frac{d\kappa}{d\varepsilon} = -\frac{\partial W}{\partial K} = -\sum_{j=0}^{n-1} \frac{\varepsilon^j}{j!} \left(\frac{n-j}{n+1} \right) K^{\frac{-1-j}{n+1}} \tilde{F}_j(\kappa, t)$$

Now let $C = \max |\tilde{F}_j(\kappa, t)|$ and with $\sum_{j=0}^{n-1} \frac{\varepsilon^j}{j!} \left(\frac{n-j}{n+1} \right) < B$ we have

$$-BC < \frac{d\kappa}{d\varepsilon} < BC$$

so that also $\kappa(\varepsilon)$ exists up until $\varepsilon = 1$. □

In order to account for the time dependency we must compute the remainder function R, which is the transform of $-\partial W / \partial t$. Since W only involves $p_j(t)$ for $j = n, \ldots, 2n-2$ they must be at least C^1 so far, but $p_j(t)$ for $j = 0, \ldots, n-1$ have not yet appeared in W so that they only need to be continuous for the first transformation.

More specifically we have to transform

$$-\frac{\partial W}{\partial t} = -\sum_{j=0}^{n-2} \frac{\varepsilon^j}{j!} \frac{\partial W_{j+1}}{\partial t}$$

via another Lie triangle using the same generating function W. If the entries for that triangle are denoted by R_j^i and

$$R_j^0 = -\frac{\partial W_{j+1}}{\partial t} = -K^{\frac{n-j}{n+1}} \frac{\partial \tilde{F}_j}{\partial t}(\kappa, t) \qquad \text{for} \quad j = 0, 1, \ldots$$

then the remainder function is

$$R(K, \kappa, t, \varepsilon) = \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!} R_0^i(K, \kappa, t).$$

The construction of R follows the same argument as given above with two exceptions. First W is already known with it's entries in $\mathcal{R}_j$ and second the beginning entries are R_j^0 are also in $\mathcal{R}_j$ not in $\mathcal{P}_j$. Thus all the entries of the triangle, R_{j-i}^i are in $\mathcal{R}_j$. That means that

$$R_0^j = K^{\frac{n-j}{n+1}} \tilde{G}_j(\kappa, t) \quad \text{for } j = 0, 1, \ldots, \infty \tag{14}$$

where $\tilde{G}_j(\kappa, t)$ is another periodic function.

Thus at the end of the first transformation we arrive at the intermediate Hamiltonian

$$\mathcal{H}_1 = H_0^0 + \sum_{i=1}^{\infty} \frac{1}{i!}(H_0^i + R_0^{i-1}).$$

The terms for H_0^i are given in (11)–(13). Since R_0^{i-1} contains $K^{\frac{n-i}{n+1}}$ it is added to those of (13) when terms with the same powers of K are combined. On the other hand combining the terms does not change those in (12). Thus we arrive at the form which was given in (5).

5.2 The Second Transformation

This time we will transform $\mathcal{H}_1$ to $\mathcal{H}$. So consider

$$H_*(K, \kappa, t, \varepsilon) = \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!} H_i^0(K, \kappa, t) \tag{15}$$

where now

$$\begin{aligned} H_0^0 &= \tfrac{n+1}{2n} K^{\frac{2n}{n+1}}, \\ H_i^0 &= i! K^{\frac{2n-i}{n+1}} \bar{f}_i(t), \qquad \text{for } i = 1, \ldots, n, \\ H_i^0 &= i! K^{\frac{2n-i}{n+1}} f_i(\kappa, t), \quad \text{for } i = n+1, \ldots. \end{aligned} \tag{16}$$

The intermediate Hamiltonian $\mathcal{H}_1$ is obtained from (15) by setting $\varepsilon = 1$. Since the first n rows already have the desired form we set $W_r = 0$ for $r = 1, \ldots, n$ and determine W_r for $r = n+1, \ldots, 2n$ so that the terms H_0^r do not depend on κ. We also set $W_r = 0$ for $r = 2n+1, \ldots$, so that Lemma 1 can be used which shows that also this transformation is convergent for $\varepsilon = 1$. The remainder is calculated as before and with it we find

$$\mathcal{H} = H_0^0 + \sum_{i=1}^{\infty} \frac{1}{i!}(H_0^i + R_0^{i-1}).$$

Finally by grouping terms with the same powers of K we see that we have obtained $\mathcal{H}$ in the form as displayed in (4).

The first transformation removed the κ dependences for terms in rows $r = 1, \ldots, n-1$ of the Lie triangle, and the remainder function required that the $p_j(t)$, $j = n, \ldots, 2n-2$ be C^1. The second transformation removed the κ dependences for terms which appear in rows $r = n, \ldots, 2n-2$. It required that the $p_j(t)$, $j = 1, \ldots, n-1$ be C^1, but the $p_j(t)$ for $j = n, \ldots, 2n-2$ also appeared in the new generating function W so that they must be C^2 in total. However $p_0(t)$ did not occur in either of the generating functions so that it needs only to be C^0.

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