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NORMALLY STABLE HAMILTONIAN SYSTEMS

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To Ernesto Lacomba on his 65th Birthday

Abstract. We study the stability of an equilibrium point of a Hamiltonian system with n degrees of freedom. A new concept of stability called normal stability is given which applies to a system in normal form and relies on the existence of a formal integral whose quadratic part is positive definite. We give a necessary and sufficient condition for normal stability. This condition depends only on the quadratic terms of the Hamiltonian. We relate normal stability with formal stability and Liapunov stability. An application to the stability of the L_4 and L_5 equilibrium points of the spatial circular restricted three body problem is given.

1. **Introduction.** We give a stability criterion for an equilibrium point, the origin, of an autonomous Hamiltonian systems with n degrees of freedom which depends only on the quadratic terms of the Hamiltonian or equivalently just on the linear terms of the equations of motion. One classical results of this type is Dirichlet's Theorem [6] which states that the origin is stable when the quadratic part of the Hamiltonian is positive definite. Another classical result is that the origin is formally stable if the linear system is semi-simple, the eigenvalues are pure imaginary and are not rationally related [20]. Our criterion lies between these two classical criteria and includes them as extremes cases. We obtain a type of formal stability and is not related to the stability results in KAM theory.

This study introduces a new concept of stability called normal stability which applies to systems in normal form and relies on the existence of a formal integral whose quadratic part is positive definite. Our approach also introduces a new condition on the quadratic part of the Hamiltonian which we call the Moser-Weinstein condition since this condition is implicit in the paper of Moser [18] which reproves the classical result of Weinstein [23] on the existence of periodic solutions. The main result of this paper is that a Hamiltonian system in normal form is normally stable if and only if the Moser-Weinstein condition holds.

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We also establish the relationships between normal stability, Liapunov stability and formal stability. Indeed we prove that if the Moser-Weinstein's condition is violated one can give an analytic Hamiltonian function in normal form of n degrees of freedom such that the origin in unstable in the Liapunov sense. The construction of such a system is a slight generalization of Cherry's example [4] of a stable linear Hamiltonian system with two degrees of freedom that becomes unstable when adding a specific nonlinear term. The question whether a normally stable Hamiltonian system is necessarily Liapunov stable remains open.

Concerning formal stability we prove using the Moser-Weinstein's condition that indeed normal stability guarantees formal stability, while the converse is not true in general and we provide examples of formally stable Hamiltonians that are not normally stable. As a consequence normally stable Hamiltonians are a class of formally stable Hamiltonians. Thus, the estimates studied by Moser [17] and Glimm [9] can be applied to Hamiltonian systems that are normally stable.

Liapunov or nonlinear stability in the strong sense was initiated by A. Liapunov [12], whereas formal stability was started by Siegel [20] and Moser [15, 16] who established conditions on the quadratic terms of the Hamiltonians to achieve formal stability. Glimm [9] and Bryuno [2] generalized the works of Siegel and Moser in order to assure formal stability by taking into consideration the quartic terms in normal form. Khazin [10] and Kunitsyn and Tuyakbayev [11] dealt with systems with n degrees of freedom, enlarging previous results.

A related concept is Birkhoff stability (also called Lie stability) introduced by Khazin [10], see also [19], which analyzes the stability in the sense of Liapunov of a Hamiltonian in normal form starting at a certain degree, say $\mathcal{H}_m$. More precisely, after fixing a value $m > 2$, it studies if the normal form Hamiltonian truncated at order j is stable for all $j \geq m$. This type of stability is different from normal stability.

The paper is structured in seven sections. In Section 2 we set up the notation used for linear autonomous Hamiltonian systems and discuss the different types of linear stability of the origin. Section 3 is devoted to the concept of normal stability and its relationship with Liapunov stability. We establish a characterization criterion of normal stability in terms of the quadratic part of the Hamiltonian and the eigenvalues of the linearized system. We also prove that if a Hamiltonian does not satisfy this criterion one can construct a Hamiltonian in normal form for which the origin is an unstable point. The purpose of Section 4 is the study of formal stability of autonomous Hamiltonian systems and its connection with normal stability. In Section 5 we give a precise account of the theorems on the existence of periodic solutions of Moser and Weinstein that suggested this work. In Section 6 the normal stability of the three dimensional restricted three body problem at the equilibria points L_4 and L_5 is studied. We end with some concluding remarks in Section 7.

2. **Linear Hamiltonian systems.**

Consider the quadratic Hamiltonian

$$\mathbb{H}(z) = \frac{1}{2} z^T S z, \tag{1}$$

and the corresponding linear Hamiltonian system

$$\dot{z} = Az, \tag{2}$$

where $z \in \mathbb{R}^{2n}$, $\dot{z} = dz/dt$, S is a $2n \times 2n$ real symmetric matrix, $A = JS$ is a $2n \times 2n$ real Hamiltonian matrix and J is the usual $2n \times 2n$ matrix of Hamiltonian theory, i.e.

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

with I the $n \times n$ identity matrix and 0 the $n \times n$ zero matrix. The reader can consult [14] where background material with similar notation may be found.

The system of linear Hamiltonian equations (2) is *stable* if all solutions of (2) are bounded for all $t \in \mathbb{R}$, i.e. $\|e^{At}\|$ is uniformly bounded. In Hamiltonian theory stability refers to both positive and negative t. For linear systems bounded is equivalent to the usual ε-δ definition of stability.

If all the eigenvalues of A are pure imaginary and A is diagonalizable (over the complex numbers) we will say that A satisfies the pure imaginary-diagonalizable condition or the PIDC.

Theorem 2.1. *The linear Hamiltonian system (2) is stable if and only if the PIDC holds.*

If A satisfies the PIDC then one can choose real symplectic coordinates $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$\mathbb{H}(x, y) = \frac{\omega_1}{2}(x_1^2 + y_1^2) + \frac{\omega_2}{2}(x_2^2 + y_2^2) + \cdots + \frac{\omega_n}{2}(x_n^2 + y_n^2) \tag{3}$$

where the eigenvalues of A are $\pm\omega_1 i, \pm\omega_2 i, \ldots, \pm\omega_n i$ or one can choose action-angle variables

$$I_j = \frac{1}{2}(x_j^2 + y_j^2), \qquad \phi_j = \tan^{-1}\frac{y_j}{x_j}, \qquad \text{for } j = 1, \ldots, n \tag{4}$$

so that

$$\mathbb{H}(I, \phi) = \omega_1 I_1 + \omega_2 I_2 + \cdots + \omega_n I_n. \tag{5}$$

If S is a definite matrix, either positive definite or negative definite, then Dirichlet's Theorem implies that the system (2) is stable. If S is definite all the ω_j's in (3) and (5) are of the same sign. But S definite also implies that a small linear Hamiltonian perturbation is stable. This leads to the following concept and theorem.

The linear Hamiltonian system (2) is *parametrically stable* or *strongly stable* if it and all sufficiently small linear Hamiltonian perturbations of it are stable. That is, (2) is parametrically stable if there is an $\epsilon > 0$ such that $\dot{z} = Bz$ is stable, where B is any linear Hamiltonian matrix with $\|B - A\| < \epsilon$.

So by Dirichlet's Theorem $2\mathbb{H}_+ = (x_1^2 + y_1^2) + (x_2^2 + y_2^2)$ is parametrically stable. But $2\mathbb{H}_- = (x_1^2 + y_1^2) - (x_2^2 + y_2^2)$ is not parametrically stable since the small perturbation $2\mathbb{H}_\varepsilon = (x_1^2 + y_1^2) - (x_2^2 + y_2^2) + 2\varepsilon y_1 y_2$ leads to eigenvalues $\pm\sqrt{-1 \pm \varepsilon i}$.

Define $\eta(\lambda) = \text{kernel } (A - \lambda I) = \{z \in \mathbb{C}^{2n} : (A - \lambda I)z = 0\}$ to be the complex eigenspace corresponding to an eigenvalue λ. The space $\mathbb{W}_\lambda = \eta(\lambda) \oplus \eta(\bar{\lambda})$ satisfies the reality condition: if $w \in \mathbb{W}_\lambda$ then so is $\bar{w} \in \mathbb{W}_\lambda$ and therefore $\mathbb{W}_\lambda$ is the complexification of a real space $\mathbb{V}_\lambda$. The restriction of A to $\mathbb{V}_\lambda$ has eigenvalues λ, $\bar{\lambda}$.

Let A have distinct eigenvalues $\pm\beta_1 i, \ldots, \pm\beta_s i$, $1 \le s \le n$. The space $\mathbb{W}_j = \eta(\beta_j i) \oplus \eta(-\beta_j i)$ is the complexification of a real space $\mathbb{V}_j$ of dimension $2n_j$ and $n_1 + n_2 + \cdots + n_s = n$. Let A restricted to $\mathbb{V}_j$ be denoted by A_j, then $\mathbb{V}_j$ is a symplectic linear space and A_j is a real diagonalizable Hamiltonian matrix with eigenvalues $\pm\beta_j i$. Define the symmetric matrix S_j by $A_j = JS_j$ and $\mathbb{H}_j$ the restriction of $\mathbb{H}$ to $\mathbb{V}_j$.

We will say that the linear system (2) satisfies the *Krein-Gel'fand-Lidskii condition*, KGLC, if A is nonsingular, A is stable, and the Hamiltonian $\mathbb{H}_j$ is positive or negative definite for each $j = 1, \ldots, s$.

So if $\mathbb{H}_j$ is definite there are symplectic action-angle coordinates so that

$$\mathbb{H}_j = \beta_j (I_{j1} + I_{j2} + \cdots + I_{jn_j})$$

where $\beta_j > 0$ or $\beta_j < 0$.

Theorem 2.2. *The linear Hamiltonian system (2) is parametrically stable if and only if KGLC holds.*

So $\mathbb{H} = 2I_1 - I_2$ and $\mathbb{H} = I_1 + I_2$ are parametrically stable whereas $\mathbb{H} = I_1 - I_2$ is not. For the proof see pages 78–83 of [14] or consult [26] and the references therein.

3. Normally stable systems.

For the stability of an equilibrium point of a nonlinear system there are very few good results other than Dirichlet's Theorem [6] for systems of n degrees of freedom and Arnold's Theorem for systems of two degrees of freedom, see [3, 14] and references therein. Therefore we will concentrate in this section on formal systems, i.e. systems where the Hamiltonian is a formal power series in z. We are particularly interested in stability criteria which only depend on A or $\mathbb{H}$ and not the higher order terms. Since we are extending the concept of stability we shall assume throughout this section that $\mathbb{H}$ satisfies at least KGLC and so is parametrically stable. Let

$$\mathcal{H}_f(z) = \mathbb{H}(z) + \sum_{j=3}^{\infty} \mathcal{H}_j(z)$$

be a formal power series in z with real coefficients, where $\mathcal{H}_j$ is a homogeneous polynomial of degree j.

Using action-angle variables (4) leads to a Poisson series for $\mathcal{H}_f$ whose terms are of the form

$$c\, I_1^{\alpha_1/2} \cdots I_n^{\alpha_n/2} \cos(\beta_1\phi_1 + \cdots + \beta_n\phi_n)$$

where c is a real constant, the α_j's are nonnegative integers and the β_j's integers. There is also a similar sin term. Since the Poisson series came from a real power series the terms must have the d'Alembert character, i.e.

$$\alpha_j \geq |\beta_j| \quad \text{and} \quad \alpha_j \equiv \beta_j \mod 2.$$

Formal systems arise often when one puts an analytic system into normal form because in general the transformation to normal form does not converge. A formal system $\mathcal{H}_n$ is in *normal form* [13] if

$$\mathcal{H}_n(z) = \mathbb{H}(z) + \bar{\mathcal{H}}(z) \tag{6}$$

where

$$\bar{\mathcal{H}}(e^{At}z) \equiv \bar{\mathcal{H}}(z) \text{ for all } t \in \mathbb{R} \text{ and for all } z \in \mathbb{R}^{2n} \tag{7}$$

or equivalently

$$\{\mathbb{H}, \bar{\mathcal{H}}\} = 0. \tag{8}$$

Here $\{\cdot, \cdot\}$ is the Poisson bracket operator.

Group the eigenvalues of A into r groups as follows:

$$\begin{array}{l} \pm\omega_1 k_{11} i,\ \pm\omega_1 k_{12} i,\ \ldots,\ \pm\omega_1 k_{1s_1} i, \\ \pm\omega_2 k_{21} i,\ \pm\omega_2 k_{22} i,\ \ldots,\ \pm\omega_2 k_{2s_2} i, \\ \vdots \\ \pm\omega_r k_{r1} i,\ \pm\omega_r k_{r2} i,\ \ldots,\ \pm\omega_r k_{rs_r} i, \end{array} \tag{9}$$

where $\omega_1, \ldots, \omega_r$ are rationally independent and $k_{11}, \ldots, k_{rs_r}$ are nonzero integers.

For example the eigenvalues might fall into 3 groups

$$\begin{array}{llll} \pm 3i,\ \pm 7i & \text{where} & \omega_1 = 1, & k_{11} = 3,\ k_{12} = 7, \\ \pm\sqrt{2}i & \text{where} & \omega_2 = \sqrt{2}, & k_{21} = 1, \\ \pm 2\pi i,\ \pm 3\pi i,\ \pm 7\pi i & \text{where} & \omega_3 = \pi, & k_{31} = 2,\ k_{32} = 3,\ k_{33} = 7. \end{array}$$

Let $\mathbb{W}_j = [\eta(\omega_j k_{j1} i) \oplus \eta(-\omega_j k_{j1} i)] \oplus \cdots \oplus [\eta(\omega_j k_{j\sigma} i) \oplus \eta(-\omega_j k_{j\sigma} i)]$. Here we write σ for s_j to avoid double subscripts. Again $\mathbb{W}_j$ satisfies the reality condition that $w \in \mathbb{W}_j$ if and only if $\bar{w} \in \mathbb{W}_j$ so it is the complexification of a real A-invariant symplectic subspace $\mathbb{V}_j$ and

$$\mathbb{R}^{2n} = \mathbb{V}_1 \oplus \mathbb{V}_2 \oplus \cdots \oplus \mathbb{V}_r.$$

Let A_j be the restriction of A to the subspace $\mathbb{V}_j$ and $\mathbb{H}_j$ be the restriction of $\mathbb{H}$ to $\mathbb{V}_j$. Let A_j have eigenvalues

$$\pm\omega_j k_{j1} i,\ \pm\omega_j k_{j2} i,\ \ldots,\ \pm\omega_j k_{j\sigma} i.$$

We will say that the linear Hamiltonian system (2) satisfies the *Moser-Weinstein condition*, MWC, if each $\mathbb{H}_j$ is either positive or negative definite. This condition was used implicitly in [18] to detect periodic solutions, see the discussion in Section 5. Note that MWC is stronger than KGLC.

We can write the Hamiltonian $\mathbb{H}$ in the form

$$\mathbb{H} = \omega_1(k_{11} I_{11} + \cdots + k_{1s_1} I_{1s_1}) + \cdots + \omega_r(k_{r1} I_{r1} + \cdots + k_{rs_r} I_{rs_r}). \tag{10}$$

The linear Hamiltonian system (2) satisfies MWC if and only if all the $k_{\alpha\beta}$ can be chosen as positive integers and then

$$\mathbb{H}_j = \omega_j(k_{j1} I_{j1} + k_{j2} I_{j2} + \cdots + k_{js_j} I_{j\sigma})$$

is positive or negative definite as ω_j is positive or negative. First we show by an example that MWC is necessary if one wishes to determine stability from the quadratic terms alone.

Theorem 3.1. *If the linear Hamiltonian system (2) does not satisfy MWC then there is a Hamiltonian $\mathcal{H}_e(z) = \mathbb{H}(z) + \mathcal{H}_p(z)$ in normal form where $\mathcal{H}_p(z)$ is a polynomial (hence convergent) and the origin is unstable in the sense of Liapunov.*

Proof. We generalize Cherry's example as presented in [14]. If $\mathbb{H}_e$ does not satisfy MWC then it can be put into the form

$$\mathbb{H}_e(I, \phi) = \omega(k_1 I_1 - k_2 I_2) + \nu_3 I_3 + \cdots + \nu_n I_n$$

where $\omega \neq 0$, $k_1 > 0$, $k_2 > 0$, $\nu_3, \ldots, \nu_n$ are nonzero. Since we are assuming parametric stability $k_1 \neq k_2$. By changing time we set $\omega = 1$.

Let $\mathcal{H}_p(I, \phi) = I_1^{k_2/2} I_2^{k_1/2} \cos(k_2\phi_1 + k_1\phi_2)$. Note that $\mathcal{H}_p$ has the d'Alembert character and would be a polynomial when written in the z coordinates. Since $\mathcal{H}_p$

only contains I_1, I_2, ϕ_1, ϕ_2 it follows that $I_3 = \cdots = I_n = 0$ is an invariant subspace because $\dot{I}_3 = \cdots = \dot{I}_n = 0$ there. Thus we put our attention in the Hamiltonian

$$\mathcal{H}_e(I_1, I_2, \phi_1, \phi_2) = k_1 I_1 - k_2 I_2 + I_1^{k_2/2} I_2^{k_1/2} \cos(k_2\phi_1 + k_1\phi_2).$$

To see that the origin is unstable, consider the Chetaev function

$$C(I_1, I_2, \phi_1, \phi_2) = -2 I_1^{k_2/2} I_2^{k_1/2} \sin(k_2\phi_1 + k_1\phi_2)$$

and compute

$$\dot{C} = k_1^2 I_1^{k_2} I_2^{k_1 - 1} + k_2^2 I_1^{k_2 - 1} I_2^{k_1}.$$

Let Ω be the region where $C > 0$ and so $I_1, I_2 \neq 0$. But $\dot{C} > 0$ in Ω and Ω has points arbitrarily close to the origin, so Chetaev's Theorem [5, 14] shows that the origin is unstable.

Note that if we did not assume KGLC then $k_1 = k_2 = 1$ is a possibility and the perturbation $\mathcal{H}_p$ would be quadratic. In that case for a truly higher order perturbation use the term $\mathcal{H}_p(I, \phi) = I_1 I_2 \cos(2\phi_1 + 2\phi_2)$ which expressed in rectangular coordinates is a homogeneous quartic polynomial and follow the above argument *mutatis mutandis*. □

Remark: We have proved that a Hamiltonian system which does not satisfy MWC can lead to Liapunov instability, however it does not mean that a Hamiltonian of the form given in (6) whose quadratic terms do not satisfy MWC is unstable. For instance we can take $\mathcal{H} = I_1 - I_2 + I_1^7$ which is stable in the Liapunov sense but does not satisfy MWC. However, given a Hamiltonian of the form (6) whose quadratic terms satisfy MWC, the question of the stability of the origin remains open. Therefore we consider a weaker form of stability.

We will say that the linear Hamiltonian system (2) is *normally stable* if for every $\bar{\mathcal{H}}$ satisfying (7) there exists a formal integral of the form

$$\mathcal{L}_n(z) = \mathbb{L}(z) + \mathcal{L}_*(z),$$

for the Hamiltonian system $\mathcal{H}_n = \mathbb{H} + \bar{\mathcal{H}}$, where $\mathbb{L}$ is a positive definite quadratic form in z and $\mathcal{L}_*$ is a formal power series in z such that $\mathcal{L}_n$ is constant along the solutions of (2) meaning in this case $\{\mathcal{L}_n, \mathcal{H}_n\} = 0$. Of course if all the series converged the origin would be a stable equilibrium point for the system with Hamiltonian $\mathcal{H}_n$ by Dirichlet's Theorem and its Liapunov generalizations [12].

Example 1:

$$\mathbb{H} = I_1 - 2I_2$$

satisfies KGLC so is parametrically stable but does not satisfy MWC and it can lead to instability as the above theorem shows.

Example 2:

$$\mathbb{H} = I_1 - \sqrt{2} I_2$$

satisfies MWC and is normally stable. To see that it is normally stable note that the normal form has no angles so $\mathcal{L} = I_1 + \sqrt{2} I_2$ is a positive definite integral for any system in normal form starting with $\mathbb{H}$.

Example 3:

$$\mathbb{H} = I_1 + 2I_2 - \sqrt{2}(I_3 + 3I_4).$$

This $\mathbb{H}$ satisfies MWC with $\omega_1 = 1$ and $\omega_2 = \sqrt{2}$. A typical term in the normal form looks like

$$\mathcal{H} = I_1 + 2I_2 - \sqrt{2}(I_3 + 3I_4) + I_1 I_2^{1/2} I_3^{3/2} I_4^{1/2} \cos(-2\theta_1 + \theta_2 - 3\theta_3 + \theta_4).$$

Positive definite integrals for this system are

$$\mathcal{L} = I_1 + 2I_2 + \sqrt{2}(I_3 + 3I_4)$$

and

$$\mathcal{L} = I_1 + 2I_2 + \sqrt{2}(I_3 + 3I_4) + I_1 I_2^{1/2} I_3^{3/2} I_4^{1/2} \cos(-2\theta_1 + \theta_2 - 3\theta_3 + \theta_4).$$

Lemma 3.2. *Let*

$$\mathcal{T} = c\, I_{11}^{\alpha_{11}/2} \cdots I_{r\sigma}^{\alpha_{r\sigma}/2} \cos\left(\sum_{j=1}^{r} (\beta_{j1}\phi_{j1} + \cdots + \beta_{j\sigma}\phi_{j\sigma})\right)$$

be a typical term in the Poisson series for $\bar{\mathcal{H}}$ then

$$k_{j1}\beta_{j1} + k_{j2}\beta_{j2} + \cdots + k_{j\sigma}\beta_{j\sigma} = 0, \tag{11}$$

for $j = 1, \ldots, r$.

Proof. Since $\mathcal{T}$ comes from $\bar{\mathcal{H}}$ it must be constant on the solutions defined by $\mathbb{H}$. A solution is $I_{jl} = 1$, $\phi_{jl} = -\omega_j k_{jl} t$. Thus $\cos(\sum_{j=1}^{r} \omega_j (\beta_{j1} k_{j1} + \cdots + \beta_{j\sigma} k_{j\sigma})t)$ must be constant which implies $\sum_{j=1}^{r} \omega_j (\beta_{j1} k_{j1} + \cdots + \beta_{j\sigma} k_{j\sigma}) = 0$. Since the ω_j's are rationally independent (11) must hold for $j = 1, \ldots, r$. □

Theorem 3.3. *The linear Hamiltonian system (2) is normally stable if and only if MWC holds.*

Proof. Let MWC hold so $\mathbb{H}$ is of the form (10) with all the $k_{jl} > 0$. Define

$$\mathbb{L} = |\omega_1|(k_{11}I_{11} + \cdots + k_{1s_1}I_{1s_1}) + \cdots + |\omega_r|(k_{r1}I_{r1} + \cdots + k_{rs_r}I_{rs_r})$$

so $\mathbb{L}$ is positive definite. Take $\mathcal{L}_n = \mathbb{L}$ or $\mathcal{L}_n = \mathbb{L} + \bar{\mathcal{H}}$. We must show that

$$\{\mathcal{L}_n, \mathcal{H}_n\} = \{\mathbb{L} + \bar{\mathcal{H}}, \mathbb{H} + \bar{\mathcal{H}}\} = \{\mathbb{L}, \bar{\mathcal{H}}\} = 0 \tag{12}$$

and we need only look term by term. Take the term

$$\mathbb{L}^* = |\omega_j|(k_{j1}I_{j1} + k_{j2}I_{j2} + \cdots + k_{j\sigma}I_{j\sigma}).$$

Here again we write σ for the s_j's to avoid double subscripts. Since $\omega_j \neq 0$ we may cancel it out of equation (12) and forget it.

Since $\mathbb{L}^*$ depends only on the actions $I_{j1}, I_{j2}, \ldots, I_{j\sigma}$ and equation (12) is a Poisson bracket equation we need only worry about the angles $\phi_{j1}, \phi_{j2}, \ldots, \phi_{j\sigma}$ in the $\bar{\mathcal{H}}$ term. Consider the term

$$\bar{\mathcal{H}}^* = c\mathcal{I}\, I_{j1}^{\alpha_{j1}/2} I_{j2}^{\alpha_{j2}/2} \cdots I_{j\sigma}^{\alpha_{j\sigma}/2} \cos(\beta_{j1}\phi_{j1} + \beta_{j2}\phi_{j2} + \cdots + \beta_{j\sigma}\phi_{j\sigma} + \Phi)$$

where c is a constant, $\mathcal{I}$ is the product of all the other I_i's to various powers and Φ is a linear sum of all the other ϕ_i's, thus c, $\mathcal{I}$ and Φ are just constants for this computation. There is a similar sin term.

Using (11) we compute

$$\begin{aligned} \{\mathbb{L}^*, \bar{\mathcal{H}}^*\} &= (k_{j1}\beta_{j1} + \cdots + k_{j\sigma}\beta_{j\sigma}) \\ &\quad \times |\omega_j|\, c\mathcal{I}\, I_{j1}^{\alpha_{j1}/2} \cdots I_{j\sigma}^{\alpha_{j\sigma}/2} \sin(\beta_{j1}\phi_{j1} + \cdots + \beta_{j\sigma}\phi_{j\sigma} + \Phi) \\ &= 0. \end{aligned}$$

This completes the proof of that MWC implies normal stability.

If $\mathbb{H}_e$ does not satisfy MWC regardless of the number of positive and negative $k_{\alpha\beta}$ associated with $\mathbb{H}_e$ it can be put into the form

$$\mathbb{H}_e(I, \phi) = \omega(k_1 I_1 - k_2 I_2) + \nu_3 I_3 + \cdots + \nu_n I_n$$

where $\omega \neq 0$, $k_1 > 0$, $k_2 > 0$, $\nu_3, \ldots, \nu_n$ are nonzero. Also $k_1 \neq k_2$ since we are assuming KGLC. By changing time we can also assume $\omega = 1$, $\gcd(k_1, k_2) = 1$.

Let $\mathcal{H}_p(I, \phi) = I_1^{k_2/2} I_2^{k_1/2} \cos(k_2\phi_1 + k_1\phi_2)$. As $\mathcal{H}_p$ has the d'Alembert character it is a polynomial of degree $k_1 + k_2 > 2$ when it is written in the z coordinates. Therefore it is enough to look at the particular case

$$\begin{aligned}\mathcal{H}_e(I, \phi) &= \mathbb{H}_e(I, \phi) + \mathcal{H}_p(I_1, I_2, \phi_1, \phi_2)\\ &= k_1 I_1 - k_2 I_2 + \nu_3 I_3 + \cdots + \nu_n I_n + I_1^{k_2/2} I_2^{k_1/2} \cos(k_2\phi_1 + k_1\phi_2).\end{aligned}$$

To see that the origin is not normally stable, we have to prove that there is no formal integral $\mathcal{L}_e = \mathbb{L}_e + \mathcal{L}_p$ of $\mathcal{H}_e$ with $\mathbb{L}_e$ positive or negative definite. When trying to obtain $\mathcal{L}_e$, we observe that $\mathbb{L}_e$ is quadratic in the z coordinates whereas we need to choose $\mathcal{L}_p$ such that expressed in z it is a polynomial of degree $k_1 + k_2$. Then we take into account that $\{\mathcal{H}_e, \mathcal{L}_e\} = 0$ if and only if $\{\mathbb{H}_e, \mathbb{L}_e\} = 0$, $\{\mathcal{H}_p, \mathbb{L}_e\} + \{\mathbb{H}_e, \mathcal{L}_p\} = 0$ and $\{\mathcal{H}_p, \mathcal{L}_p\} = 0$. The reason for this is that $\{\mathbb{H}_e, \mathbb{L}_e\}$, $\{\mathcal{H}_p, \mathbb{L}_e\} + \{\mathbb{H}_e, \mathcal{L}_p\}$ and $\{\mathcal{H}_p, \mathcal{L}_p\}$ are homogeneous polynomials in z of respective degrees two, $k_1 + k_2$ and $2(k_1 + k_2 - 1)$.

Recall that $\{\mathbb{H}_e, \mathbb{L}_e\} = 0$ implies that $\mathbb{L}_e$ is constant along the solutions of $\mathbb{H}_e$. One solution is

$$\begin{aligned}&I_1 = 1, \quad \theta_1 = -k_1 t,\\ &I_2 = 1, \quad \theta_2 = k_2 t,\\ &I_j = 0, \quad \theta_j = \nu_j t \qquad \text{for } j = 3, \ldots, n.\end{aligned}$$

The term of lowest degree that contains the angles ϕ_1 and ϕ_2 and is constant along this solution would be $c\, I_1^{k_2/2} I_2^{k_1/2} \cos(k_2\phi_1 + k_1\phi_2)$ (or a similar sin term) which is of degree $k_1 + k_2 > 2$. Such a term would not be in $\mathbb{L}_e$. Thus

$$\mathbb{L}_e = \alpha_1 I_1 + \alpha_2 I_2 + \mathcal{F}(I_1, \ldots, I_n, \phi_1, \ldots, \phi_n)$$

for some real numbers α_1, α_2 with $\alpha_1\alpha_2 > 0$ and $\mathcal{F}$ is such that

$$\mathcal{F} = \frac{\partial \mathcal{F}}{\partial I_1} = \frac{\partial \mathcal{F}}{\partial I_2} = \frac{\partial \mathcal{F}}{\partial \phi_1} = \frac{\partial \mathcal{F}}{\partial \phi_2} = 0, \text{ when } I_3 = I_4 = \cdots = I_n = 0.$$

When expressed in the z coordinates $\mathcal{F}$ is a polynomial of degree two since it has the d'Alembert character. Indeed $\mathcal{F}$ is a linear combination of terms of the form

$$c\, I_1^{q_1/2} \cdots I_n^{q_n/2} \Big(\gamma_1 \cos(r_1\phi_1 + \cdots + r_n\phi_n) + \gamma_2 \sin(r_1\phi_1 + \cdots + r_n\phi_n)\Big)$$

with the q_i's nonnegative integers, the r_j's integers and c and the γ_j's real quantities. Since $\mathbb{L}_e$ is quadratic $q_1 + q_2 + q_3 + \cdots + q_n = 2$, but with $q_1 + q_2 < 2$ by the above condition on $\mathcal{F}$. Besides, these constants are chosen in such a way that $\mathbb{L}_e$ is a definite function in the z coordinates.

The computation of $\{\mathcal{H}_p, \mathbb{L}_e\}$ yields

$$\{\mathcal{H}_p, \mathbb{L}_e\} = -(k_2\alpha_1 + k_1\alpha_2) I_1^{k_2/2} I_2^{k_1/2} \sin(k_2\phi_1 + k_1\phi_2)$$

when $I_3 = I_4 = \cdots = I_n = 0$. Note that $k_2\alpha_1 + k_1\alpha_2$ is either positive or negative but not zero, therefore $\{\mathcal{H}_p, \mathbb{L}_e\}$ does not vanish.

On the other hand if we pick a typical pair of terms in $\mathcal{L}_p$, say $\mathcal{T}_p$, with

$$\begin{aligned}\mathcal{T}_p \;=\; I_1^{m_1/2} I_2^{m_2/2} I_3^{m_3/2} \cdots I_n^{m_n/2} \Big(&\beta_1 \cos(p_1\phi_1 + p_2\phi_2 + p_3\phi_3 + \cdots + p_n\phi_n)\\ &+ \beta_2 \sin(p_1\phi_1 + p_2\phi_2 + p_3\phi_3 + \cdots + p_n\phi_n)\Big)\end{aligned}$$

with the m_j's nonnegative integers and the p_j's integers such that they satisfy d'Alembert character in a way that if $\mathcal{T}_p$ is written in terms of z it is a polynomial of degree $k_1 + k_2$. Besides β_1, β_2 are real coefficients and together with the m_j's and the p_j's have to be determined. We get

$$\begin{aligned}\{\mathbb{H}_e, \mathcal{T}_p\} \;=\; & (k_2p_2 - k_1p_1 + p_3\nu_3 + \cdots + p_n\nu_n) I_1^{m_1/2} I_2^{m_2/2} I_3^{m_3/2} \cdots I_n^{m_n/2} \\ & \times \Big(\beta_2 \cos(p_1\phi_1 + p_2\phi_2 + p_3\phi_3 + \cdots + p_n\phi_n) \\ & \quad - \beta_1 \sin(p_1\phi_1 + p_2\phi_2 + p_3\phi_3 + \cdots + p_n\phi_n)\Big).\end{aligned}$$

In order to cancel the terms of $\{\mathcal{H}_p, \mathbb{L}_e\}$ with those of $\{\mathbb{H}_e, \mathcal{T}_p\}$ we have to choose $\beta_2 = 0$, $m_1 = p_1 = k_2$, $m_2 = p_2 = k_1$ and $p_j = m_j = 0$ for $j = 3, \ldots, n$, but then $\{\mathbb{H}_e, \mathcal{T}_p\} = 0$ and $\{\mathcal{H}_p, \mathbb{L}_e\} \neq 0$. Thus, one cannot pick the constants β_j's, m_j's, n_j's in such a way that one makes $\{\mathcal{H}_p, \mathbb{L}_e\} + \{\mathbb{H}_e, \mathcal{L}_p\} = 0$ and this is regardless of possible resonant relations.

Thus, the conclusion is that it is not possible to obtain $\mathcal{L}_p$ in order to construct $\mathcal{L}_e$ as a formal integral of $\mathbb{H}_e$ with $\mathbb{L}_e$ definite and normal stability is not established, which is a contradiction. Hence, MWC is guaranteed if normal stability holds. □

4. **Formal stability.** Consider a real analytic Hamiltonian

$$\mathcal{H}_a(w) = \mathbb{H}(w) + \sum_{j=3}^{\infty} \mathcal{H}_j(w) \tag{13}$$

such that the origin is an equilibrium point and the $\mathcal{H}_j$'s are homogeneous polynomials of degree j in $w \in \mathbb{R}^{2n}$.

The analytic system (13) is *formally stable* [2, 15, 16] if there exists a formal series $\mathcal{L}_f(w)$ which is positive definite and a formal integral for (13), i.e. $\{\mathcal{H}_a, \mathcal{L}_f\} = 0$.

Theorem 4.1. *If the linear Hamiltonian system (2) satisfies MWC then the analytic Hamiltonian system (13) is formally stable, thus normal stability implies formal stability.*

Remark: MWC is not necessary for formal stability. For instance $\mathcal{H}_a = I_1 - 2I_2 + I_1^7$ is not normally stable but is formally stable — $\mathcal{L}_f = I_1 + I_2$ is a positive definite integral. Furthermore higher order terms in $\mathcal{L}_f$ can assure the positive definiteness, see for example [10] and [22], where conditions on the higher order terms are given so that one obtains formally stable resonant Hamiltonians (5) with a single resonance of order four but no resonance of order three. Hence, the quadratic part of this nonlinear Hamiltonian could be $\mathbb{H} = 3I_1 - I_2$ while higher order terms can be chosen so that to get a formally stable system or an unstable one.

However, if one demands for formal stability that the formal integral has a positive definite quadratic part then MWC would be necessary.

Proof. Since $\mathbb{H}$ satisfies the PIDC there is a near identity formal symplectic change of variables $w = W(z)$ that transforms $\mathcal{H}_a(w)$ in (13) to normal form $\mathcal{H}_n(z)$ in (6), see [14]. That is $\mathcal{H}_a(W(z)) = \mathcal{H}_n(z)$.

Let $\mathcal{L}_n(z)$ be the positive definite integral constructed in the last section (in Theorem 3.3). The function $W(z)$ has an inverse $z = Z(w)$ which is also a near identity formal symplectic change of variables. Let $\mathcal{L}_f(w) = \mathcal{L}_n(Z(w))$. Now since symplectic transformations preserve Poisson brackets

$$\{\mathcal{H}_a, \mathcal{L}_f\} = \{\mathcal{H}_n, \mathcal{L}_n\} = 0.$$

Thus $\mathcal{L}_f(w)$ is a positive definite formal integral for $\mathcal{H}_a$. □

Remark: Concerning the issue of Hamiltonian systems where normal stability is easy to check while formal stability is harder, we need to pick an example of more than three degrees of freedom so that to avoid single resonances. According to [19] (Theorem 3.1, p. 812) assuming that a linear system (2) has a single resonance with annihilating vector $(k_1, \cdots, k_n)$ (e.g. $k_1\omega_1 + \cdots + k_n\omega_n = 0$) if there exists $k_i \neq k_j$ with $k_i k_j < 0$ then the corresponding equilibrium is Birkhoff stable and formally stable. If we consider the example

$$\mathbb{H} = \sqrt{2}(I_1 + I_2 + I_3) - \sqrt{3}(I_4 + 2I_5),$$

then MWC is satisfied and normal stability (and then formal stability) holds for any perturbation in normal form one adds. However formal stability of a Hamiltonian whose quadratic terms are like those given by $\mathbb{H}$ is not straightforwardly obtained from the linear vector field itself as the system does not possess a single resonance.

5. **Moser-Weinstein Theorem.** Now that all the notation has been given we can explain Moser's extension to Weinstein's Theorem on the existence of periodic solutions to a Hamiltonian system and how it lead to this work. They consider an analytic or just a smooth system

$$\mathcal{H}_s(z) = \mathbb{H}(z) + \mathcal{H}_h(z) \tag{14}$$

where as before $\mathcal{H}_h$ represents terms higher than quadratic in z. If A is nonsingular and the linear system (2) satisfies the PIDC then all solutions of (2) are periodic with periods $2\pi/|\omega_1|, \ldots, 2\pi/|\omega_n|$.

In a series of papers [23, 24, 25] Weinstein proved the following.

Theorem 5.1. *If $\mathbb{H}(z)$ is positive definite then the system (14) has n periodic solutions on the energy surface $\mathcal{H} = \varepsilon$ for $\varepsilon > 0$ and small. These solutions have periods close to the periods of the linear system (2).*

Moser gives an alternate proof of this theorem in [18]. If we use the notation of this paper the following is one of the consequences of his proof. Assume that the linear system satisfies MWC and so the eigenvalues can be grouped as in (9). Then all the solutions of the linear system $\dot{z}(z) = A_j z$ on $\mathbb{V}_j$ are periodic with period $2\pi/|\omega_j|$ (not necessarily the least period). Now $\mathbb{H}_j$ is positive or negative definite as ω_j is positive or negative. Theorem 4 of [18] asserts that there are n_j periodic orbits of period $2\pi/|\omega_j|$ near $\mathbb{V}_j$ when $\mathcal{H} = \varepsilon > 0$ if $\mathbb{H}_j$ is positive definite or on $\mathcal{H} = \varepsilon < 0$ if $\mathbb{H}_j$ is negative definite. Thus a corollary of Moser's Theorem 4 is the following result.

Theorem 5.2. *If $\mathbb{H}(z)$ satisfies MWC then the system (14) has n periodic solutions on the energy surfaces $\mathcal{H} = \pm\varepsilon$ small. These solutions have periods close to the periods of the linear system (2).*

6. **Application.** A linear Hamiltonian system with three degrees of freedom satisfying the PIDC has a quadratic Hamiltonian of the form $\mathbb{H} = \omega_1 I_1 + \omega_2 I_2 + \omega_3 I_3$. In order to discuss if MWC holds for $\mathbb{H}$ for various values of the ω_i's there are two basic possibilities, either (i) the three frequencies ω_i's all have the same sign or (ii) two of them have one sign and the other one has the opposite sign. Without loss of generality one can suppose that: (i) $\omega_i > 0$ for all i or (ii) $\omega_1 < 0$, $\omega_2 > 0$ and $\omega_3 > 0$. In case (i) MWC is always guaranteed while in (ii) MWC holds if and only if the quotients ω_2/ω_1 and ω_3/ω_1 are irrational. In the traditional criteria for formal

stability in case (ii) one would be required on to check that there is no relation of the form $r_1\omega_1 + r_2\omega_2 + r_3\omega_3 = 0$ where $r_1, r_2, r_3 \in \mathbb{Z}$ and not all zero.

On the other hand a linear Hamiltonian system with four degrees of freedom satisfying the PIDC has a quadratic Hamiltonian of the form $\mathbb{H} = \omega_1 I_1 + \omega_2 I_2 + \omega_3 I_3 + \omega_4 I_4$ and there are three possibilities to check for MWC. Without loss of generality the cases are: (i) $\omega_i > 0$ for all i, (ii) $\omega_i > 0$ for $i = 1, 2, 3$ and $\omega_4 < 0$, and (iii) $\omega_1 > 0$, $\omega_2 > 0$ and $\omega_3 < 0$, $\omega_4 < 0$. In case (i) MWC is always satisfied whereas MWC holds in (ii) if and only if the quotients ω_i/ω_4 for $i = 1, 2, 3$ are all irrational and MWC holds in (iii) if and only if ω_1/ω_3, ω_2/ω_3, ω_1/ω_4 and ω_2/ω_4 are all irrational. The discussion follows similarly for more degrees of freedom with an increasing number of possibilities.

Now we apply the theory of this paper to the study of the normal and formal stability of the equilibrium points L_4 and L_5 of the circular restricted three body problem in the spatial case. The problem deals with the motion of an infinitesimal particle subject to the gravitational influence of two massive particles with masses m_1 and m_2 in the three dimensional space [21]. The Hamiltonian written in a rotating frame $x_1x_2x_3$ is given by

$$\begin{aligned}\mathcal{H}_R &= \tfrac{1}{2}(y_1^2 + y_2^2 + y_3^2) - (x_1y_2 - x_2y_1)\\ &\quad - \frac{\mu}{\sqrt{(x_1 - 1 + \mu)^2 + x_2^2 + x_3^2}} - \frac{1-\mu}{\sqrt{(x_1 + \mu)^2 + x_2^2 + x_3^2}}.\end{aligned}$$

The parameter μ stands for the quotient $m_1/(m_1+m_2)$ and assuming that $m_1 \geq m_2$ then μ is in $(0, 1/2)$. The masses m_1 and m_2 are located at the points $(-\mu, 0, 0)$ and $(1 - \mu, 0, 0)$ of the coordinate space, respectively.

The Hamiltonian system $\mathcal{H}_R$ has five equilibria, the Euler points L_i, $i = 1, 2, 3$ and the Lagrangian points L_i, $i = 4, 5$. The points L_1, L_2 and L_3 lie on the axis x_1 while the coordinates of L_4 and L_5 are $(1/2 - \mu, \sqrt{3}/2, 0)$ and $(1/2 - \mu, -\sqrt{3}/2, 0)$ respectively.

The points L_1, L_2 and L_3 are unstable of saddle-center type, thus from now on we focus on the stability character of L_4 and L_5. While the stability in the planar problem ($x_3 = y_3 = 0$) has been widely studied, see for instance [14], the spatial case is harder and the Liapunov stability issue remains an open question. Taking into account the values of the momenta for these points, in the six dimensional phase space the coordinates of L_4 and L_5 are $(1/2 - \mu, \pm\sqrt{3}/2, 0, \mp\sqrt{3}/2, 1/2 - \mu, 0)$ (the upper signs for L_4 and the lower for L_5). Shifting the origin to L_4 (or to L_5), linearizing $\mathcal{H}_R$ around the origin, giving the same name to the coordinates and dropping the constant terms, the resulting Hamiltonian is

$$\mathcal{H}_R = \frac{1}{2}(y_1^2 + y_2^2 + y_3^2) - (x_1y_2 - x_2y_1) + \frac{1}{8}x_1^2 \pm \frac{3\sqrt{3}(2\mu - 1)}{4}x_1x_2 - \frac{5}{8}x_2^2 + \frac{1}{2}x_3^2 + \cdots .$$

The associated eigenvalues are $\pm\lambda_1$, $\pm\lambda_2$ and $\pm\lambda_3$ with

$$\lambda_1 = \frac{\sqrt{-1 - \sqrt{27\mu^2 - 27\mu + 1}}}{\sqrt{2}},\ \lambda_2 = \frac{\sqrt{-1 + \sqrt{27\mu^2 - 27\mu + 1}}}{\sqrt{2}},\ \lambda_3 = i.$$

When $\mu > \mu_R = \frac{1}{2}(1 - \sqrt{69}/9)$, i.e. the Routh's value, the equilibria are of focus-center type, therefore unstable as they come from a symplectic system, so we restrict μ to $(0, \mu_R]$. In this interval the eigenvalues λ_i are all pure imaginary. Moreover in $(0, \mu_R)$ the corresponding eigenvectors form a basis of $\mathbb{R}^6$, thus the linear system satisfies the PIDC. We introduce the frequencies $\omega_j = -\lambda_j i$, hence

$\omega_j > 0$ for $j = 1, 2, 3$. Note that $0 < \omega_2 < \sqrt{2}/2 < \omega_1 < 1$, $\omega_1^2 + \omega_2^2 = 1$ and $\omega_3 = 1$. Following similar steps to those of [14] (pp. 73–75) although increasing the dimension by two due to the appearance of x_3 and y_3, we build a symplectic change that transform the quadratic terms of the Hamiltonian to

$$\mathbb{H}_R = -\frac{\omega_1}{2}(y_1^2 + x_1^2) + \frac{\omega_2}{2}(y_2^2 + x_2^2) + \frac{\omega_3}{2}(y_3^2 + x_3^2), \tag{15}$$

where we have kept the same name for the transformed coordinates. We notice that $\mathbb{H}_R$ is indefinite. Changing to action-angle coordinates we end up with

$$\mathbb{H}_R = -\omega_1 I_1 + \omega_2 I_2 + \omega_3 I_3.$$

Various authors [1, 8] have focused on Nekhoroshev stability of L_4 and L_5 for the spatial problem when $\mu \in (0, \mu_R)$. Nekhoroshev stability is a notion weaker than Liapunov stability, that instead of asking for stability for all positive time, one asks for stability for exponentially long times. More precisely, Nekhoroshev stability at a fixed point means that $d(0) \le \epsilon$ implies $d(t) \le \epsilon^a$ for $|t| \le \exp(\epsilon^{-b})$ for some positive constants a, b, where $d(t)$ is the distance of a point $z(t)$ of a given orbit to the fixed point. In [8] the authors prove Nekhoroshev stability excepting for a few values of μ that lead to resonances.

We are in case (ii) of the first paragraph of this section. Thus, to check if MWC holds it is enough to ensure that the quotients ω_2/ω_1 and ω_3/ω_1 are irrational numbers. We observe that $0 < \omega_2/\omega_1 < 1$ and $1 < \omega_3/\omega_1 < \sqrt{2}$. There are infinite values of μ such that $\omega_2/\omega_1 = r \in \mathbb{Q}$ with $r \in (0, 1)$ and $\omega_3/\omega_1 = 1/s \in \mathbb{Q}$ with $s \in (1/\sqrt{2}, 1)$ given by

$$\mu_r = \frac{1}{2} - \frac{\sqrt{27r^4 + 38r^2 + 27}}{6\sqrt{3}(r^2 + 1)}, \quad \mu_s = \frac{1}{2} - \frac{\sqrt{48s^4 - 48s^2 + 81}}{18}. \tag{16}$$

Proposition 1. *The Lagrange points L_4 and L_5 are normally stable points in the spatial circular restricted three body problem for all values of $\mu \in (0, \mu_R)$ excepting for $\mu = \mu_r$ and $\mu = \mu_s$ given in (16) with r and s rational numbers such that $r \in (0, 1)$ and $s \in (1/\sqrt{2}, 1)$.*

We remark that ω_1 and ω_2 are rationally independent excluding the value $\mu = \mu_r$ while ω_1 and ω_3 are rationally independent excluding the value $\mu = \mu_s$. The three frequencies can be rationally dependent when $\mu_r = \mu_s$ and s and r rational numbers. It occurs for $s = 1/\sqrt{r^2 + 1}$, $r = m/n$ with m and n integers such that $\sqrt{m^2 + n^2}$ is also an integer. For instance, if $m = 3$ and $n = 4$ then $r = 3/4$, $s = 4/5$ and $(\omega_1, \omega_2, \omega_3) = (3/5, 4/5, 1)$ but if $m = 5$ and $n = 12$ then $r = 5/12$, $s = 12/13$ and $(\omega_1, \omega_2, \omega_3) = (12/13, 5/13, 1)$.

Finally, we can apply Theorem 5.2 of Section 5 to the equilibria L_4 and L_5 concluding that there are three families of periodic solutions around L_4 (and around L_5) when MWC holds, that is for $\mu \in (0, \mu_R)$ and $\mu \ne \mu_r$, $\mu \ne \mu_s$ with r and s rational numbers in $(0, 1)$ and $(1/\sqrt{2}, 1)$ respectively. These periodic solutions have periods near $2\pi/\omega_i$, $i = 1, 2, 3$ and are the so called Liapunov orbits (horizontal short and long periodic families and vertical family) obtained using Liapunov Center Theorem, see for instance [14].

7. **Conclusions.**

We introduce a new type of stability for an equilibrium point of an autonomous Hamiltonian system with n degrees of freedom, namely normal stability. We provide a straightforward criterion to decide the normal stability of a certain equilibrium simply from the linearized vector field associated with the

Hamiltonian expanded around the equilibrium. We prove that normal stability is a special type of formal stability. We relate the concept of normal stability with the existence of n periodic solutions for systems that satisfy MWC. These periodic solutions have periods close to the periods of the linear systems. We apply the theory to the study of the stability of the Lagrangian points of the spatial circular restricted three body problem for values of the parameter μ in the interval $(0, \mu_R)$.

Other applications for systems with four degrees or more of freedom can be studied in the context of normal stability. Then, formal stability is more difficult to be checked as there are no theorems available for dealing with formally stable Hamiltonians with non-single resonances, which are common if $n > 3$, and then higher order terms of the normal form Hamiltonians should be analyzed. Thus in some cases normal stability would be treated more efficiently. An example of this is the stability analysis of the equilibria in Riemann ellipsoids [7].

Normal stability can be enlarged to deal with time periodic Hamiltonian systems, where the quadratic terms $\mathbb{H}$ are autonomous. If they depend on time Floquet-Liapunov theory [14] can be applied to build a symplectic linear transformation so that the resulting linear system be time independent. Thus, the case of normal stability of periodic solutions could be tackled.

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